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Unit 4 Notes / Secondary 3 Honors

Day 1: Operations on Rational Functions

Rational Functions: A function which can be written in the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions. $q(x) \neq 0$.

fractions :)

Examples:

$$f(x) = \frac{1}{x} \qquad g(x) = \frac{x}{x+1} \qquad h(x) = \frac{x^2+1}{x-3}$$

$$p(x) = \frac{x+2}{x^2+1} \qquad q(x) = \frac{3x^2-4x+1}{x^2+x-2} \qquad r(x) = \frac{4x^3+7x^2-3}{(x-2)(x+4)}$$

Simplifying Rational Expressions: - divide out common FACTORS

* if any "+" or "-" are outside parentheses ... Do NOT cross things off!

- **Restrictions:** Any values of x that make the denom = 0. (* restrictions are found BEFORE you simplify.)

Examples: Simplify the following expressions. State any domain restrictions.

1. $\frac{x^2-x-2}{x+1}$ $x \neq -1$

$= \frac{(x-2)\cancel{(x+1)}}{\cancel{(x+1)}} = \boxed{x-2}$

2. $\frac{x^2-x-2}{x^2+2x+1}$

$= \frac{(x-2)\cancel{(x+1)}}{(x+1)\cancel{(x+1)}}$

$= \boxed{\frac{(x-2)}{(x+1)}}$ $x \neq -1$

3. $\frac{\cancel{(x-2)}\cancel{(x+2)}(x+4)^2(x-4)^2}{\cancel{(x-2)}(x+2)^2\cancel{(x+4)}}$

$= \boxed{\frac{(x+4)(x-4)^2}{(x+2)}}$ $x \neq 2, -2, -4$

4. $\frac{8x^3-4x^2}{2x^2}$

$= \frac{\cancel{4}^{2}\cancel{x^2}(2x-1)}{\cancel{2x^2}}$

$= \boxed{2(2x-1)}$ $x \neq 0$

5. $\frac{a^2+5a+4}{a^2+9a+20}$

$= \frac{\cancel{(a+4)}(a+1)}{\cancel{(a+4)}(a+5)}$

$= \boxed{\frac{(a+1)}{(a+5)}}$ $x \neq -4, -5$

6. $\frac{2a^2+10a}{3a^2+15a}$

$= \frac{2\cancel{a}\cancel{(a+5)}}{3\cancel{a}\cancel{(a+5)}}$

$= \boxed{\frac{2}{3}}$ $a \neq 0, -5$

7. $\frac{48x^5y^3}{6x^2y^4}$

$= \boxed{\frac{8x^3}{y}}$ $x \neq 0, y \neq 0$

Multiplying and Dividing Rational Expressions:

Mult = straight across → then simplify

Divide = Mult by reciprocal → then simplify

****Remember you can simplify FACTORS when multiplying or dividing.**

8. $\frac{x+5}{x^2-4x+3} \cdot \frac{x-3}{4x+20}$

$= \frac{(\cancel{x+5})(\cancel{x-3})}{(\cancel{x-3})(x-1)(4)(\cancel{x+5})}$

$= \boxed{\frac{1}{4(x-1)}}$

9. $\frac{3x^2+15x}{x^2-3x-40} \div \frac{5x^2}{x^2-64}$

$= \frac{3\cancel{x}(\cancel{x+5})(x+8)(\cancel{x-8})}{(\cancel{x-8})(\cancel{x+5})(5)(x)(\cancel{x})}$

$= \boxed{\frac{3(x+8)}{5x}}$

10. $\frac{8-7x-x^2}{x+8} \cdot \frac{x+5}{9x-9}$

$= \frac{-(x^2+7x-8)(x+5)}{(x+8)(9)(x-1)}$

$= -\frac{(\cancel{x+8})(\cancel{x-1})(x+5)}{(\cancel{x+8})(9)(\cancel{x-1})} = \boxed{\frac{-(x+5)}{9}}$

11. $\frac{x^2-16}{9-x} \cdot \frac{x^2+x-90}{x^2+14x+40}$

$= \frac{(\cancel{x+4})(x-4)(\cancel{x+10})(\cancel{x-9})}{-(\cancel{x-9})(\cancel{x+10})(\cancel{x+4})}$

$= \boxed{-(x-4)}$

12. $\frac{10x^2-20x}{40x^3-80x^2} \div \frac{6x+30}{16x^3+80x^2}$

$= \frac{\cancel{10}x(\cancel{x-2})(\cancel{16}^{4\,2}\cancel{x^2})(\cancel{x+5})}{\cancel{40}_{4}\cancel{x^2}(\cancel{x-2})(\cancel{6}_{3})(\cancel{x+5})}$

$= \boxed{\frac{2x}{3}}$

13. $\frac{5x^2y}{x^7} \div \frac{30xy^4}{y^3}$

$= \frac{5x^2y \cdot y^3}{x^7 \cdot 30xy^4} = \frac{\cancel{5}x^2\cancel{y^4}}{\cancel{30}_{6}x^8\cancel{y^4}}$

$= \boxed{\frac{1}{6x^6}}$

Day 2: **Operations on Rational Functions**

Adding and Subtracting Rational Expressions:

* Common denominator

* then combine tops — simplify answer if possible

****Remember you must have a common denominator to add or subtract fractions.**

Examples: Find the LCD.

1. $2x^2$ and $4x+12$

$2xx \quad 4(x+3)$

$LCD = 2xx\,2(x+3) = \boxed{4x^2(x+3)}$

2. $2x-5$ and $2x$

$(2x-5) \quad (2)(x)$

$LCD = (2x-5)(2)(x) = \boxed{2x(2x-5)}$

3. x^2-9 and $x+3$

$(x+3)(x-3) \quad (x+3)$

$LCD = \boxed{(x+3)(x-3)}$

4. x^2-16 and $x^2-8x+16$

$(x+4)(x-4) \quad (x-4)(x-4)$

$LCD = \boxed{(x+4)(x-4)(x-4)}$

Examples: Perform the given operation and simplify.

5. $\dfrac{2}{x-1}+\dfrac{3}{x-1} = \boxed{\dfrac{5}{x-1}}$

6. $\dfrac{5x-6}{x^2-9}-\dfrac{4}{x-3}\cdot\dfrac{(x+3)}{(x+3)}$

LCD $(x+3)(x-3)$

$x^2-9 = (x+3)(x-3)$

$= \dfrac{5x-6}{(x+3)(x-3)} - \dfrac{4x+12}{(x+3)(x-3)} = \boxed{\dfrac{x-18}{(x+3)(x-3)}}$

7. $\dfrac{(x-5)}{(x-5)}\dfrac{x-7}{x^2-3x+2}+\dfrac{4}{x^2-7x+10}\dfrac{(x-1)}{(x-1)}$

LCD $(x-2)(x-1)(x-5)$

$(x-2)(x-1) \quad (x-5)(x-2)$

$= \dfrac{x^2-12x+35}{(x-2)(x-1)(x-5)} + \dfrac{4x-4}{(x-2)(x-1)(x-5)}$

$= \boxed{\dfrac{x^2-8x+31}{(x-2)(x-1)(x-5)}}$

8. $\dfrac{(x+1)}{(x+1)}\dfrac{x}{3x-15}-\dfrac{2x+2}{x^2-4x-5}\dfrac{3}{3}$

LCD $3(x-5)(x+1)$

$3(x-5) \quad (x-5)(x+1)$

$= \dfrac{x^2+x}{3(x-5)(x+1)} - \dfrac{6x+6}{3(x-5)(x+1)}$

$= \dfrac{x^2-5x-6}{3(x-5)(x+1)} = \dfrac{(x-6)\cancel{(x+1)}}{3(x-5)\cancel{(x+1)}} = \boxed{\dfrac{x-6}{3(x-5)}}$

9. $\dfrac{x-3}{x-3}\dfrac{x+2}{2x-2}-\dfrac{-2x-1}{x^2-4x+3}\dfrac{2}{2}$

LCD $2(x-1)(x-3)$

$2(x-1) \quad (x-3)(x-1)$

$= \dfrac{x^2-x-6}{2(x-3)(x-1)} - \dfrac{-4x-2}{2(x-3)(x-1)} = \dfrac{x^2+3x-4}{2(x-3)(x-1)} = \dfrac{(x+4)\cancel{(x-1)}}{2(x-3)\cancel{(x-1)}} = \boxed{\dfrac{(x+4)}{2(x-3)}}$

Complex Fractions:

10. $\dfrac{\frac{1}{x+2}-1 \; \frac{x+2}{x+2}}{\frac{3}{x^2-4}}$

$= \dfrac{\frac{1}{x+2} - \frac{x+2}{x+2}}{\frac{3}{x^2-4}} = \dfrac{\frac{-x-1}{x+2}}{\frac{3}{x^2-4}}$

$= \dfrac{-x-1}{\cancel{x+2}} \cdot \dfrac{(\cancel{x+2})(x-2)}{3}$

$= \boxed{\dfrac{(-x-1)(x-2)}{3}}$

11. $\dfrac{\frac{x}{2}-8 \; \frac{2}{2}}{\frac{x}{x}10+\frac{6}{x}}$

$= \dfrac{\frac{x}{2} - \frac{16}{2}}{\frac{10x}{x} + \frac{6}{x}} = \dfrac{\frac{x-16}{2}}{\frac{10x+16}{x}}$

$= \dfrac{x-16}{2} \cdot \dfrac{x}{10x+16} = \boxed{\dfrac{x(x-16)}{2(10x+16)}}$

Day 3: **Solving Rational Equations**

Method 1: **Cross Multiply** - Only if single fraction = single fraction

Solve the equations for x.

1. $\dfrac{x}{12} = \dfrac{5}{1}$

$x(1) = 12(5)$

$\boxed{x = 60}$

2. $\dfrac{x}{x+1} = \dfrac{5}{1}$

$5(x+1) = x$

$5x+5 = x$

$4x = -5$

$\boxed{x = -5/4}$

3. $\dfrac{x}{x+1} = \dfrac{5}{3}$

$5x+5 = 3x$

$2x = -5$

$\boxed{x = -5/2}$

Method 2: Multiply by the LCD. When can't cross multiply.

* Multiply both sides by LCD (this gets rid of fractions. :))
* Solve remaining equation.
* CHECK SOLUTIONS!

* **Extraneous Solutions:**

Solutions that don't really work in original equation.

LCD 12

4. $12\left[\frac{x}{3}+\frac{3x}{4}=2\right]12$

$12\cdot\frac{x}{3}+12\cdot\frac{3x}{4}=12\cdot 2$

$4x+9x=24$

$5x=24$

$\boxed{x=24/5}$ ✓

LCD $x(x-2)$

5. $\left[\frac{2}{x}=\frac{3}{x-2}-1\right]x(x-2)$

$x(x-2)\cdot\frac{2}{x}=x(x-2)\cdot\frac{3}{x-2}-x(x-2)\cdot 1$

$2x-4=3x-x^2+2x$

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$\boxed{x=4,-1}$ ✓

LCD $(x+2)(x-2)$

6. $\left[\frac{1}{x-2}=\frac{3}{x+2}-\frac{6x}{x^2-4}\right](x+2)(x-2)$

$1(x+2)=3(x-2)-6x$

$x+2=3x-6-6x$

$8=-4x$

$\boxed{\cancel{-2=x}}$ restriction

$\boxed{\text{No Solution}}$

LCD $x(x+2)$

7. $\left[\frac{x+1}{x+2}+\frac{5}{x}=\frac{2x+9}{x+2}\right]x(x+2)$

$x(x+1)+5(x+2)=x(2x+9)$

$x^2+x+5x+10=2x^2+9x$

$0=x^2+3x-10$

$0=(x+5)(x-2)$

$\boxed{x=-5,2}$ ✓ neither are restrictions :)

****What if you have a single fraction equal to zero?** Fraction is zero if top is zero & bottom isn't.

8. $0=\frac{x^2-5x-6}{x+7}$

$0=x^2-5x-6$

$=(x-6)(x+1)$

$\boxed{x=6,-1}$ neither are restrictions

9. $\frac{3x^2+15x}{x^2-3x-40}=0$

$(x-8)(x+5)$

$3x^2+15x=0$

$3x(x+5)=0$

$\boxed{x=0}$, $\cancel{-5}$ -5 is a restriction

Application of rational functions:

Melinda has decided that it is time to replace her old refrigerator. She purchases a new Energy Star certified refrigerator. Energy Star certified refrigerators use less electricity than those that are not certified. In the long run, the Energy Star refrigerator should cost Melinda less to operate.

1. Melinda purchases a new Energy Star refrigerator for $2000. The refrigerator costs $46 per year to operate.

   a. Assume that the refrigerator is reliable and its only costs of ownership are the purchase price and the cost of operation. Determine Melinda's average annual cost of owning the new refrigerator for the given number of years.

   1 year: $\frac{\$2000 + \$46}{1\text{ yr}} = \boxed{\$2046}$

   5 years: $\frac{\$2000 + \$46(5)}{5} = \boxed{\$446}$

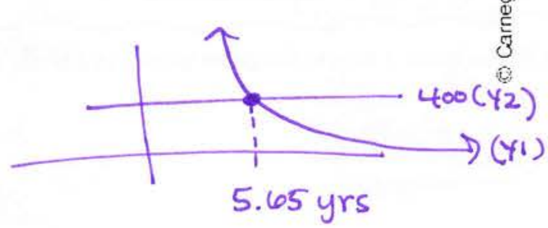
   10 years: $\frac{\$2000 + \$46(10)}{10} = \boxed{\$246}$

   b. Write an expression to represent Melinda's average annual cost of owning the new refrigerator for *x* years.

   $\frac{2000 + 4x}{x}$ $\quad x = \#$ of years

   c. When Melinda's average annual cost of owning the refrigerator is less than $400, she plans to shop for a new television. When can Melinda shop for a new television?

   $\frac{2000 + 46x}{x} < 400$



After 5.65 years

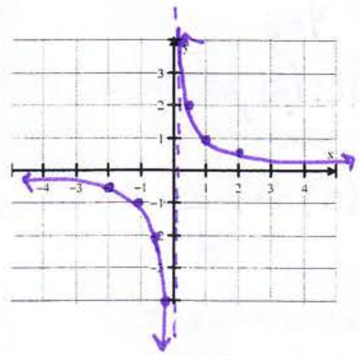
Day 4: Introduction to Graphs of Rational Functions

Complete the table of values for the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ and plot the points on the graphs.

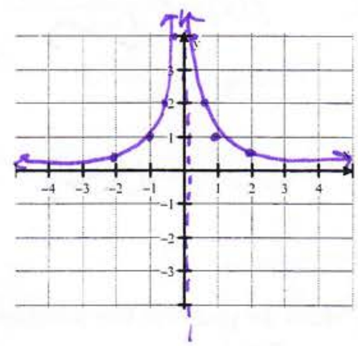
x	$f(x)$
-2	-1/2
-1	-1
-½	-2
-¼	-4
0	undef.
¼	4
½	2
1	1
2	1/2

x	$g(x)$
-2	1/4
-1	1
-½	4
-¼	16
0	undef
¼	16
½	4
1	1
2	1/4

$f(x)$



$g(x)$



Vertical Asymptotes

The line x = a is a vertical asymptote of the graph of f(x) if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ either from the right or from the left.

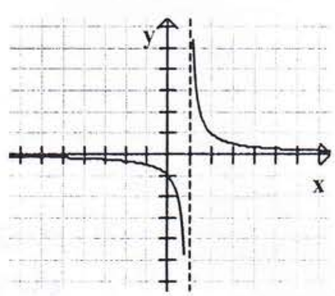
Vertical asymptotes come from: Values of x that make the denom = 0

THEY CANNOT BE CROSSED!!

***Find the vertical asymptotes for the following functions*:**

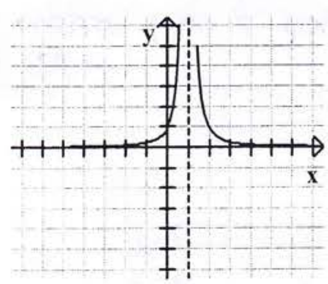
1. $y = \dfrac{1}{x-1}$ Odd vertical asymptote

VA: X = 1



2. $y = \dfrac{1}{(x-1)^2}$ Even vertical asymptote

VA: X = 1



****How does the graph behave if the vertical asymptote is odd? Is even?**

ODD => graph pulls in opposite directions on each side

EVEN => graph pulls in same direction on each side

Holes in the Graph

If a number creates a zero in both the numerator and the denominator of a rational function, it *could* create a <u>hole</u> in the graph. If the same number also creates a vertical asymptote, then there is NOT a hole.

3. $f(x) = \dfrac{\cancel{(x+1)}(x-2)}{\cancel{(x+1)}}$ Hole? Yes

$\boxed{\text{Hole: } (-1,-3)}$

$f_{red}(x) = (x-2)$ $f(-1) = -1-2$ $= -3$

4. $f(x) = \dfrac{\cancel{(x+1)}(x-2)}{(x+1)^{\cancel{2}}}$ Hole? NO

x=-1 is VA

$f_{red}(x) = \dfrac{(x-2)}{(x+1)}$

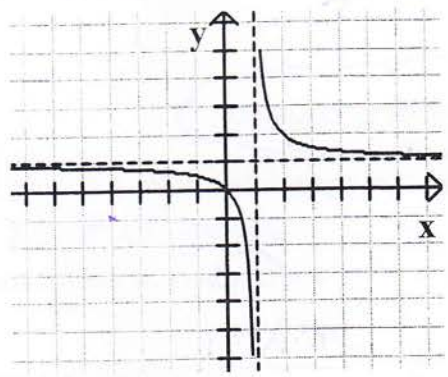
****How do you find the y-value of a hole in the graph?**

Plug x-value of hole into reduced function.

Horizontal Asymptotes:

The end behavior for a rational function What graph is approaching on right side and left side.

The line y = b is a horizontal asymptote of the graph of f(x) if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.



Horizontal asymptotes come from:

- if deg top < deg bottom HA : y=0
- if deg of top = deg bottom HA: $y = \frac{LC}{LC}$
- if deg top > deg bottom NO HA

THEY CAN BE CROSSED

Find the horizontal asymptotes:

5. $y = \dfrac{(x+1)(x+3)}{x^3}$ d=2 d=3

$\boxed{HA: y=0}$

6. $y = \dfrac{2x+3}{3x-1}$ d=1 d=1

$\boxed{HA: y=\frac{2}{3}}$

7. $y = \dfrac{(x+6)(x-2)}{(x-1)}$ d=2 d=1

$\boxed{HA: \text{none}}$

x-intercepts: **Where the graph crosses or touches the x-axis. Let y = 0.**

They are **odd** if: Show up an odd # of times – cross through

They are **even** if: Show up an even # of times – bounce point

Find the x-intercepts of the following functions:

8. $y = \frac{(x+3)(x-2)}{x^3}$

$0 = (x+3)(x-2)$

$x = -3, 2$

$\boxed{(-3,0) \quad (2,0)}$

9. $y = \frac{(x+1)^2(x+3)^3}{x^6}$

$0 = (x+1)^2(x+3)^3$

$x = -1, -3$

$\boxed{(-3,0) \quad (-1,0)}$

y-intercepts: **Where the graph crosses the y axis. Let x = 0.**

Find the y-intercepts of the following:

10. $y = \frac{(x+1)(x-2)}{(x+3)^2}$

$y = \frac{(0+1)(0-2)}{(0+3)^2}$

$y = \frac{(1)(-2)}{(3)^2}$

$y = \frac{-2}{9}$

$\boxed{(0, -2/9)}$

11. $y = \frac{(x+4)}{x^2}$

$y = \frac{(0+4)}{0^2}$ undefined

$\boxed{\text{NO y-int}}$

Day 5: Graphs of Rational Functions

Steps to analyzing graphs of rational functions without a calculator.

1. Factor both the numerator and denominator if necessary.
2. Find the domain. (This comes from the ***unreduced*** fraction.)
3. Find the holes (x , y)
4. Find the vertical asymptotes (odd or even , x =   )
5. Find the end behavior (HA, SA, other).
6. Find the y-intercept (let x = 0). At most there can be one.
7. Find the x-intercepts (let y = 0). Comes from the numerator. Odd or even?
8. If necessary, find a point or two. – only if #1-7 don't give you enough

*If a hole is on the x axis, then it has the same even nature (bounces) or odd nature (goes through) for the graph as it does an x-intercept. However, **a hole is not an x-intercept**.

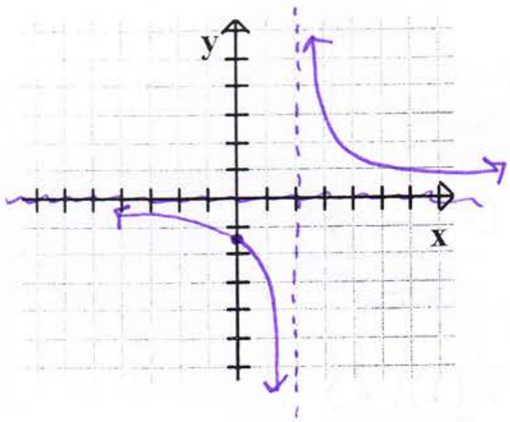
Try to graph the following without a calculator.

1. $f(x)=\dfrac{3}{x-2}$ can't reduce

Domain: $\mathbb{R}$, $x \neq 2$ y-int: $(0, -3/2)$

VA: $x=2$ (odd)   x-int: none

EB: HA: $y=0$



2. $g(x)=\dfrac{2x-1}{x^2}$ can't reduce

Domain: $\mathbb{R}$ $x \neq 0$ y-int: none

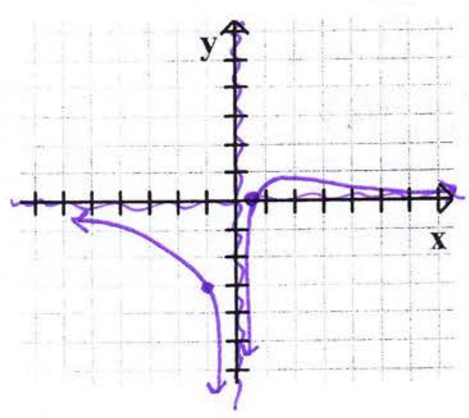
VA: $x=0$ (even)   x-int: $(1/2, 0)$

EB: HA: $y=0$

Additional point

x	y
-1	-3

$$\frac{2(-1)-1}{(-1)^2} = \frac{-3}{1} = -3$$



3. $h(x)=\frac{x^2}{x^2-x-2} = \frac{x^2}{(x-2)(x+1)}$

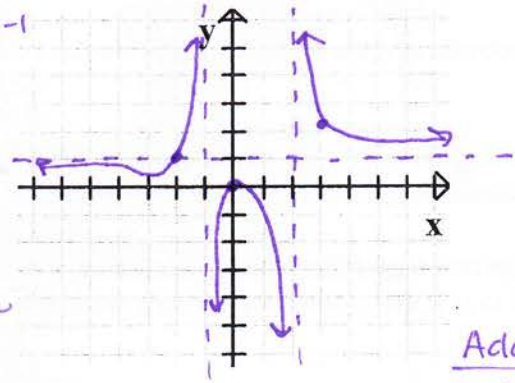
Domain: $\mathbb{R}, x \neq 2, -1$

VA: $x=2$ odd
$x=-1$ odd

HA: $y=1$

y-int: $(0,0)$

x-int: $(0,0)$ even



y-int: $y = \frac{0^2}{0^2-0-2} = \frac{0}{-2} = 0$.

x-int: $0 = x^2$
$0 = x$

Cross HA? $(y=1)$

$1 = \frac{x^2}{x^2-x-2}$

$x^2-x-2 = x^2$
$-x-2 = 0$
$-x = 2$
$x = -2$ ✓

Additional

x	y
-2	1
3	9/4

$y = \frac{3^2}{3^2-3-2} = \frac{9}{4}$

4. $k(x)=\frac{2(x^2-4)}{x^2-9} = \frac{2(x+2)(x-2)}{(x+3)(x-3)}$

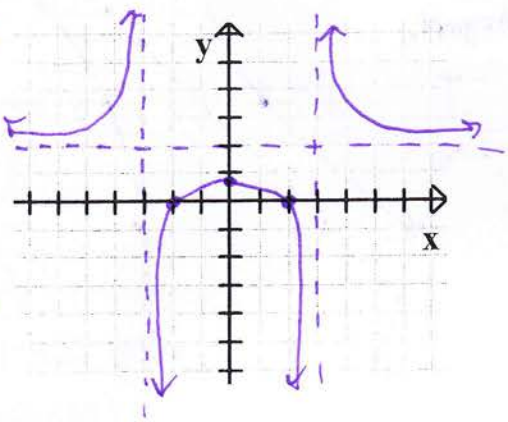
Domain: $x \neq 3, -3$

VA: $x=3$ odd
$x=-3$ odd

HA: $y=2$

y-int: $(0, 8/9)$

x-int: $(-2,0)$ odd
$(2,0)$ odd



y-int: $y = \frac{2(0+2)(0-2)}{(0+3)(0-3)} = \frac{-8}{-9} = 8/9$

x-int: $0 = 2(x+2)(x-2)$ $x=-2, 2$

Cross HA? $(y=2)$

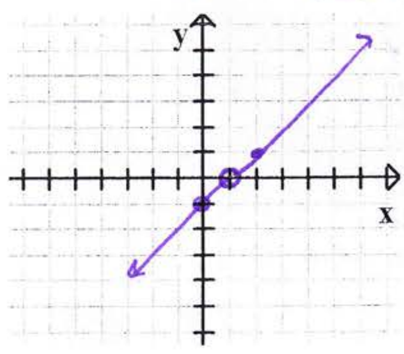
$2 = \frac{2(x^2-4)}{x^2-9}$

$2(x^2-9) = 2(x^2-4)$
$x^2-9 = x^2-4$ No
$-9 \neq -4$

5. $f(x)=\frac{(x-1)^{\cancel{2}}}{\cancel{x-1}} = x-1$ graph this w/ hole

Hole
$(1,0)$

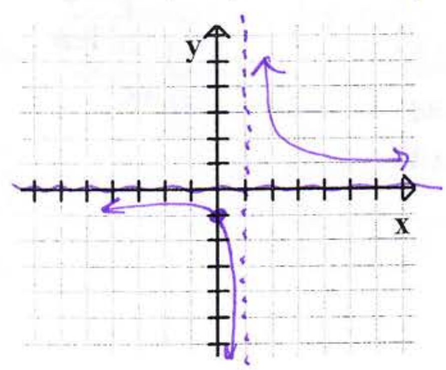
$f_r(1) = 1-1 = 0$



f(x) has a removable discontinuity
↓
Hole

6. $g(x)=\frac{(\cancel{x-1})}{(x-1)^{\cancel{2}}} = \frac{1}{x-1}$

VA: $x=1$ odd
HA: $y=0$
x-int: none
y int: $(0,-1)$
No Hole



g(x) has a non-removable discontinuity
↓
VA

Day 6: Graphs of Rational Functions

End Behavior

- If the **degree** is the **same** on the top and bottom, or if the degree is bigger on the bottom, you have a **horizontal asymptote**.
- If the degree of the **numerator** is exactly **1 bigger** than the **denominator**, you have a **slant asymptote**. To find the equation of the slant asymptote use division and ignore the remainder. diagonal
- If the degree of the **numerator** is **more than one bigger** than the **denominator,** the end behavior approximates the shape of the polynomial that would result from dividing the numerator by the denominator.

1. $f(x) = \dfrac{x^2+3x-4}{x+1} = \dfrac{(x+4)(x-1)}{(x+1)}$

VA: $x=-1$ odd

y-int: $(0,-4)$

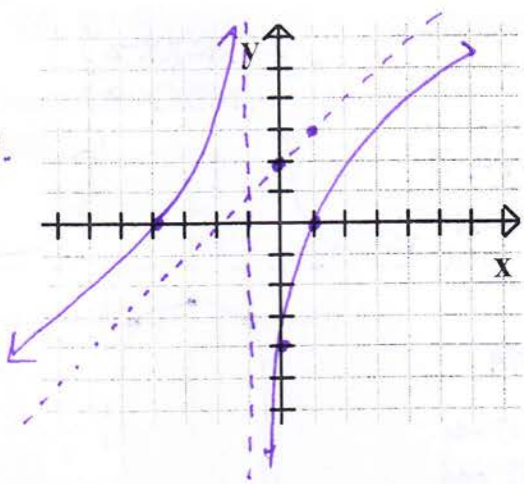
x-int: $(-4,0)$ odd
$(1,0)$ odd

SA: $y = x+2$

Slant Asympt.

$$\begin{array}{r|rrr} -1 & 1 & 3 & -4 \\ & & -1 & -2 \\ \hline & 1 & 2 & -6 \end{array}$$

$x+2$



2. $f(x) = \dfrac{x^2-x-2}{x-1} = \dfrac{(x-2)(x+1)}{(x-1)}$

VA: $x=1$ odd

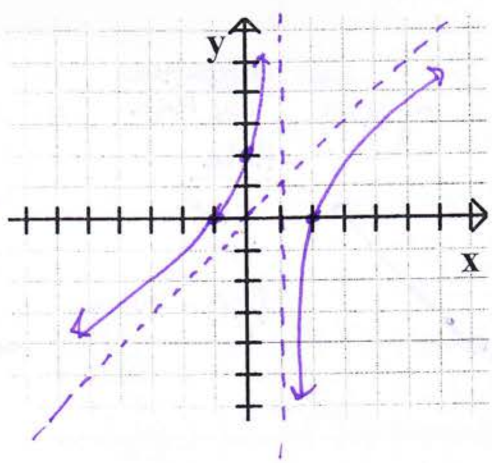
SA: $y=x$

x-int: $(2,0)$ odd
$(-1,0)$ odd

y-int: $(0,2)$

$$\begin{array}{r|rrr} 1 & 1 & -1 & -2 \\ & & 1 & 0 \\ \hline & 1 & 0 & -2 \end{array}$$

$y=x$



Graph the following without a calculator.

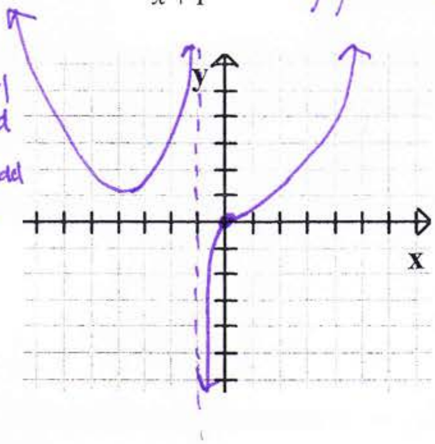
3. $f(x)=\dfrac{x^3}{x+1}$

EB ↑↑ if divide $\frac{x^3}{x}$ get $+x^2$

VA: x=-1 odd

X int (0,0) odd

y int (0,0)



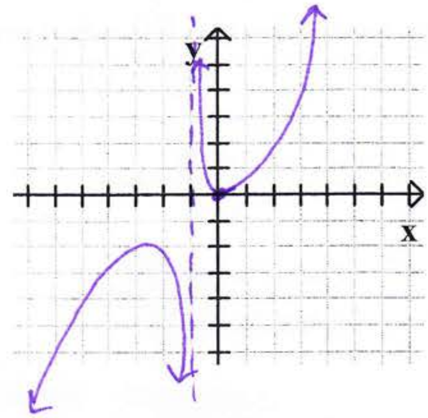
4. $f(x)=\dfrac{x^6}{x+1}$

EB ↓↑ $+x^5$

VA: x=-1 odd

Xint (0,0) even

yint (0,0)



5. $g(x)=\dfrac{-(x+4)^2(x-2)(x+3)^2}{(x+1)}$

$\frac{-x^5}{x} = -x^4$

VA: X=-1 odd

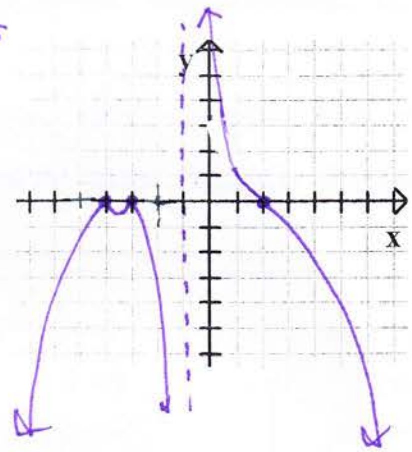
EB: ↓↓

Xint: (-4,0) even

(2,0) odd

(-3,0) even

yint: (0,288)



6. $f(x)=\dfrac{(\cancel{x-2})(\cancel{x+2})(x+4)^{\cancel{2}}(x-4)^2}{(\cancel{x-2})(x+2)^{\cancel{2}}(\cancel{x+4})}$

possible Holes x=2, ~~-2~~, -4

$f_{red}(x) = \dfrac{(x+4)(x-4)^2}{(x+2)}$

$\frac{x^3}{x} = +x^2$

VA: X=-2 odd

Holes: (2, 6)

(-4, 0) even

EB: ↑↑

xint: (4,0) even

y-int: (0,32)