Unit 5 Notes / Secondary 3 Honors

Day 1: Introduction to Limits

Limit definition: If $f(x)$ (y) becomes "arbitrarily close" (→ approaches) to some unique number L as x approaches C from either side, then the limit of $f(x)$ as x approaches C is L.

$$\boxed{\lim_{x\to c} f(x) = L}$$ "Limit as x approaches c of f(x) is L"

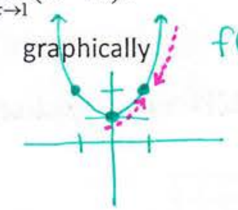
Translation: as x approaches a #, the y approaches what # ?

(Remember, sometimes you get there and sometimes you don't. It doesn't matter. You are simply looking at what the value is approaching. (getting closer to))

Finding a limit that can be reached:

1. Find $\lim_{x\to 1}(x^2+1)$:

a. graphically



$f(x)=x^2+1$

as $x \to 1$

$y \to ?$

$=\boxed{2}$

b. algebraically – substitute 1 into the function

$(1)^2+1 = 1+1 = \boxed{2}$

Oftentimes the limit of $f(x)$ as x→c is simply $f(c)$. **Substitute first!!** if you get an answer then that's the limit – if not try other things.

2. Find the limits:

a. $\lim_{x\to 4} x^2$

$= 4^2 = \boxed{16}$

b. $\lim_{x\to 4} 5$

$= \boxed{5}$

c. $\lim_{x\to 4}(x^2+2x-5)$

$= 4^2+2(4)-5$

$= 16+8-5 = \boxed{19}$

d. $\lim_{x\to 4}\frac{x^2+2x-5}{x-1}$

$= \frac{4^2+2(4)-5}{4-1} = \boxed{\frac{19}{3}}$

Finding a limit that cannot be reached:

Indeterminate Form: $\frac{0}{0}$ ... could be hole or could be VA.

3. Find $\lim_{x\to 2}\frac{x^3-2x^2+x-2}{x-2}$

If try substitution:

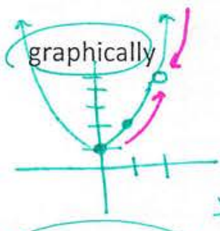
$\frac{(2)^3-2(2)^2+2-2}{2-2} = \frac{0}{0}$

a. graphically

$\frac{x^2(x-2)+(x-2)}{(x-2)}$

$x^2+1 = f(x)$

Hole: (2, 5)



as $x \to 2$:

$y \to ?$

$=\boxed{5}$

* graph in calc & trace if need to

b. by using a table – in Calc.

Tbl Set : 1.9

Δ tbl : .01

then watch the y's .... What are they getting closer to?

$\boxed{5}$

c. algebraically

*must use correct notation

$\lim_{x\to 2}\frac{x^2(x-2)+(x-2)}{(x-2)}$

$= \lim_{x\to 2}\frac{(x^2+1)(x-2)}{(x-2)} = \lim_{x\to 2} x^2+1 = 2^2+1$

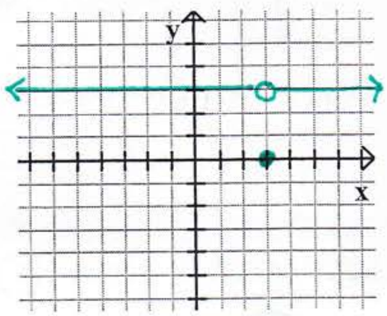
$=\boxed{5}$

4. **Use your calculator** to find $\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$ $= \boxed{2}$

* using table...
tbl start = 0
Δtbl = .01
see what y is approaching

or graph & trace
but trace isn't
as accurate sometimes

5. Graph the piecewise function $f(x)=\begin{cases}4, & x\neq 3\\ 0, & x=3\end{cases}$



Use the graph to find $\lim_{x\to 3} f(x)$.

$\lim_{x\to 3} f(x) = \boxed{4}$

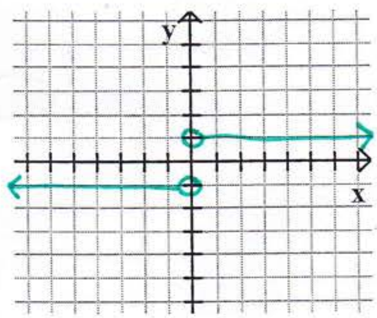
* value that y is approaching
not necessarily the value y gets to.

Note: From the examples you should see that $f(x)$ does not need to exist where x = c for a limit to exist at c.

Limits that Fail to Exist – can't get an answer

- **Different behavior from the left of c and the right of c.** – approaching different values

6. Graph $f(x)=\frac{|x|}{x}$ D: $x\neq 0$
(so no point there)



Where does the limit not exist?

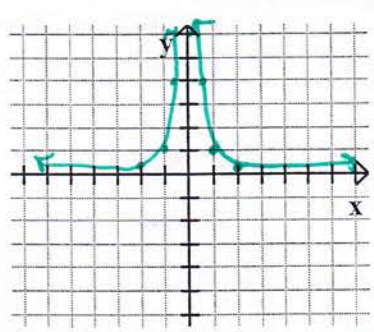
@ x=0

Why? the graph approaches -1 from the left and +1 from the right.

* jump in the graph

- **Unbounded behavior**

7. Graph $f(x)=\frac{1}{x^2}$



Where does the limit not exist?

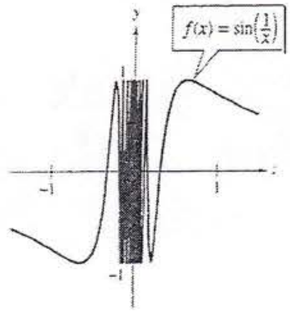
@ x=0

Why?
The y's are not approaching a number.
They are getting bigger & bigger...
* Vertical asymptote

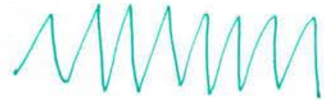
$\lim_{x\to 0}\frac{1}{x^2} = \boxed{DNE}$

- **Oscillating behavior**

Look at the graph of $f(x)=\sin\left(\frac{1}{x}\right)$ below to see why $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.



* DNE because y-values oscillate as y approaches 0.



Summary:

Limits do not exist when:

1. Jump in the graph (Not just a hole)
2. Vertical asymptote
3. Oscillating behavior

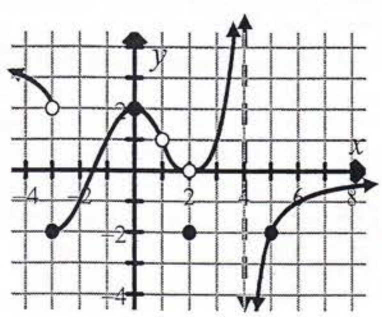
9. Find the following limits using the graph of $g(x)$.

a. $\lim_{x\to 0} g(x)$ $\boxed{2}$

b. $\lim_{x\to 4} g(x)$ $\boxed{\text{DNE}}$

c. $\lim_{x\to 1} g(x)$ $\boxed{1}$

d. $\lim_{x\to 2} g(x)$ $\boxed{0}$



* What about $\lim_{x\to -3} g(x)$?

DNE (jump in graph)

Day 2: Evaluating Limits

Limits of Polynomial and Rational Functions

Limits of polynomial and rational functions can be found by **direct substitution** if substituting does not create a zero in a denominator.

1. Find $\lim\limits_{x\to -1}\dfrac{x^2+x-6}{x+3}$ $= \dfrac{(-1)^2+(-1)-6}{(-1)+3} = \dfrac{1-1-6}{-1+3} = \dfrac{-6}{2} = \boxed{-3}$

Cancellation Method: Factor & Cancel

If substitution does create a zero in the denominator of a rational function, try to **factor** the numerator and denominator of the function, and then **cancel common factors** to reduce the function. **Substitute** into the **reduced function** to find the limit. (Do you remember $f(x)$ vs. f_{red} (x) and holes in the graph??)

2. Find $\lim\limits_{x\to -3}\dfrac{x^2+x-6}{x+3}$ Substitute first: $\dfrac{(-3)^2+(-3)-6}{-3+3} = \dfrac{0}{0}$ ∴

$f_{red}(x) = x-2$

a. algebraically:

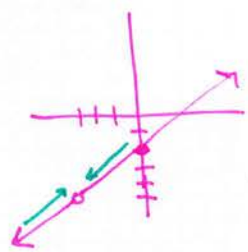
$\lim\limits_{x\to -3}\dfrac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}}$

$= \lim\limits_{x\to -3}(x-2)$

$= -3-2 = \boxed{-5}$

b. graphically:

Hole: $(-3,-5)$



$\lim\limits_{x\to -3} f(x) = \boxed{-5}$

Rationalizing Method:

Use the conjugate

A function containing radicals $\sqrt{\ }$ which produces $\frac{0}{0}$ when using direct substitution may produce a limit if its numerator and/or denominator are rationalized (multiply by a conjugate form). → use the conjugate of the part with the root.

Subst $\dfrac{\sqrt{0+1}-1}{0} = \dfrac{0}{0}$ ∴

3. Find $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}\cdot\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$= \lim\limits_{x\to 0}\dfrac{x+1-1}{x(\sqrt{x+1}+1)}$

$= \lim\limits_{x\to 0}\dfrac{x}{x(\sqrt{x+1}+1)}$

$= \lim\limits_{x\to 0}\dfrac{1}{\sqrt{x+1}+1} = \dfrac{1}{\sqrt{0+1}+1} = \boxed{\frac{1}{2}}$

Subst $\dfrac{4-4}{\sqrt{4+5}-3} = \dfrac{0}{0}$ ∴

4. Find $\lim\limits_{x\to 4}\dfrac{x-4}{\sqrt{x+5}-3}\cdot\dfrac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$= \lim\limits_{x\to 4}\dfrac{(x-4)(\sqrt{x+5}+3)}{x+5-9}$

$= \lim\limits_{x\to 4}\dfrac{\cancel{(x-4)}(\sqrt{x+5}+3)}{\cancel{(x-4)}}$

$= \lim\limits_{x\to 4}(\sqrt{x+5}+3) = \sqrt{4+5}+3$

$= \boxed{6}$

Complex Fractions: **Common Denominator & Reduce** Subst first : $\frac{\frac{1}{0+1}-1}{0} = \frac{0}{0}$

5. Find $\lim_{x\to 0}\frac{\frac{1}{x+1}-1}{x}$

$= \lim_{x\to 0}\frac{\frac{1}{x+1}-\frac{x+1}{x+1}}{x}$

$= \lim_{x\to 0}\left(\frac{-x}{x+1}\cdot\frac{1}{x}\right)$

$= \lim_{x\to 0} -\frac{1}{x+1}$

$= -\frac{1}{0+1} = \boxed{-1}$

To find limits of non-algebraic functions, you must often use more sophisticated techniques, requiring you to enlist your calculator.

Technology Method – problems will tell you when it's ok to use calculator.

Use your calculator to graph and then find values of $f(x)$ for x close to c.

6. Find $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ to 4 decimal place accuracy.

Table
tbl start = 0
Δtbl = .0001

$\boxed{2.7183}$

7. Find $\lim_{x\to 0}\frac{\sin x}{x}$ to 4 decimal place accuracy.

* Radian mode always for trig functions

$\boxed{1}$

One Sided Limits

Sometimes a limit fails to exist because approaching c from the left produces a different value of L than approaching c from the right when attempting to find $\lim_{x\to c} f(x)$. In such cases you have 2 different one-sided limits.

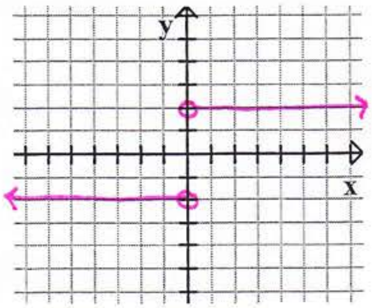
$\lim_{x\to c^-} f(x) = L_1$ represents the limit from the left (of c).

$\lim_{x\to c^+} f(x) = L_2$ represents the limit from the right (of c).

$\lim_{x\to c} f(x) \longrightarrow$ means from both L & R

8. Find $\lim_{x\to 0^-}\frac{|2x|}{x} = \boxed{-2}$ $\qquad \lim_{x\to 0^+}\frac{|2x|}{x} = \boxed{2}$

a. graphically w/ calculator



b. algebraically

plug in value left of 0

$\lim_{x\to 0^-}\frac{|2x|}{x} = \frac{|2(-1)|}{-1} = \frac{2}{-1} = \boxed{-2}$

value right of 0

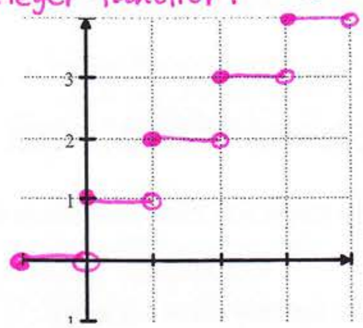
$\lim_{x\to 0^+}\frac{|2x|}{x} = \frac{|2(1)|}{(1)} = \frac{2}{1} = \boxed{2}$

9. Find with your calculator: $[x]$ = greatest integer function (in calc int(x))

$\lim_{x\to 3^+}[x+1] = \boxed{4}$

$\lim_{x\to 3^-}[x+1] = \boxed{3}$

$\lim_{x\to 3}[x+1] = \boxed{DNE}$



Existence of a Limit

If f is a function and c and L are real numbers, then $\lim_{x\to c} f(x) = L$ if and only if both the **left and right limits** are equal to L.

10. Find $\lim_{x\to 2} h(x)$ if $h(x) = \begin{cases} 2(x+1)^2 - 3, & x < 2 \\ -3x+4, & x \ge 2 \end{cases}$

a. algebraically

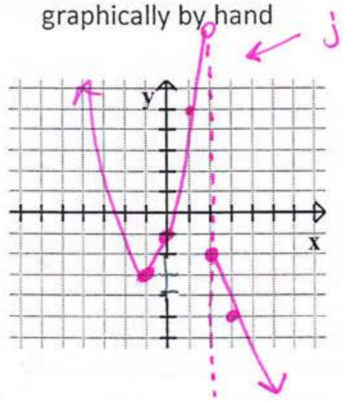
Left of 2 ($x<2$): $2(2+1)^2 - 3 = 15$

Right of 2 ($x \ge 2$): $-3(2)+4 = -6+4 = -2$

diff #'s from L & R

So, $\boxed{DNE}$

b. graphically by hand



jump in graph @ $x=2$ So $\boxed{DNE}$

11. Find $\lim_{x\to 1} f(x)$ if $f(x) = \begin{cases} 4-x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$

Left: ($x<1$) $4-1 = 3$

Right: ($x>1$) $4(1)-(1)^2 = 4-1 = 3$

same #

So $\lim_{x\to 1} f(x) = \boxed{3}$

12. An overnight delivery service charges \$8 for the first pound and \$2 for each additional pound. Let x represent the weight of a parcel and let f(x) represent the shipping cost. Show that the limit of f(x) as x→2 does not exist.

$$f(x)=\begin{cases}8, & 0<x\le 1\\ 10, & 1<x\le 2\\ 12, & 2<x\le 3\end{cases}$$

$\lim_{x\to 2^-} f(x) = 10$

$\lim_{x\to 2^+} f(x) = 12$ $\therefore$ DNE

Summary of Limits

Limits exist:

When there is a point or a hole in the graph @ that given x-value.

Limits do not exist:

1. jump in graph
2. VA $\left(\frac{\#}{0}\right)$
3. Oscillates

To find limits you can:

1. Plug in x (direct substitution)
   *if get an answer that's the limit if graph is continuous.
2. If $\frac{0}{0}$ .... need to "fix" it.
   - factor | cancel
   - use conjugate if $\sqrt{}$
   - complex fraction – simplify

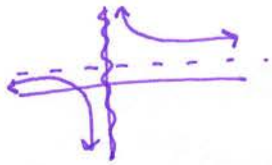
   then substitute
3. If can't simplify function on your own ... use calc (graph or table)

Day 3: Limits at Infinity

Definition: **If f is a function and L_1 and L_2 are real numbers, the statements $\lim_{x\to-\infty} f(x) = L_1$ and $\lim_{x\to\infty} f(x) = L_2$ denote the limits at infinity.**

Consider the function $f(x) = \frac{x+1}{2x}$. Earlier we discovered that the horizontal asymptote for the graph of this function is $y = \frac{1}{2}$. Why? $\left(\frac{LC}{LC} \text{ since degrees are equal} \right)$

Using limit notation, this can be written as follows:



$\lim_{x\to-\infty} f(x) = \frac{1}{2}$ (Horizontal asymptote to the left.)

$\lim_{x\to\infty} f(x) = \frac{1}{2}$ (Horizontal asymptote to the right.)

These limits (at infinity) mean that the value of $f(x)$ gets "arbitrarily" close to ½ as x decreases or increases without bound.

* Limit @ infinity is really just "End behavior." If HA, then that value is the limit. If SA or ↑↓, then DNE.

1. Find the following limits:

a. $\lim_{x\to\infty} \frac{3x+5}{x-2}$

HA: $y=3$

$\boxed{3}$

b. $\lim_{x\to\infty} \frac{7x^3 - 8x^2 + 9x}{12x^3 + 7x + 12}$

HA: $y = 7/12$

$\boxed{7/12}$

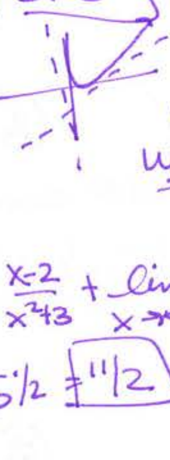
c. $\lim_{x\to\infty} \frac{5}{12x^3 + 3x^2}$

HA: $y=0$

$\boxed{0}$

d. $\lim_{x\to\infty} \frac{x^2}{x+1}$ SA

$\boxed{\text{DNE}}$



y is getting bigger and bigger unbounded

2. Find $\lim_{x\to\infty} \left(4 - \frac{3}{x^2} \right) = \boxed{4}$

HA: $y=4$ $= \lim_{x\to\infty} \left(\frac{4x^2}{x^2} - \frac{3}{x^2} \right) = \lim_{x\to\infty} \left(\frac{4x^2 - 3}{x^2} \right)$

OR

$= \lim_{x\to\infty} 4 - \lim_{x\to\infty} \frac{3}{x^2} = 4 - 0 = \boxed{4}$

3. Find $\lim_{x\to\infty} \left(\frac{7x+3}{14x+1} - \frac{x-2}{x^2+3} + 5 \right)$

$= \lim_{x\to\infty} \frac{7x+3}{14x+1} - \lim_{x\to\infty} \frac{x-2}{x^2+3} + \lim_{x\to\infty} 5$

$= \frac{1}{2} - 0 + 5 = 5\frac{1}{2} = \boxed{11/2}$

4. Find $\lim_{x\to\infty} \left(\frac{7x+3}{14x+1} - \frac{x^2 - 2}{x+3} + 5 \right)$

$= \lim_{x\to\infty} \frac{7x+3}{14x+1} - \lim_{x\to\infty} \frac{x^2-2}{x+3} + \lim_{x\to\infty} 5 = \frac{1}{2} - \underset{(\infty)}{\text{DNE}} + 5 = \boxed{\text{DNE}}$

5. Find the limit as x approaches infinity for each of the following functions:

a. $f(x)=\dfrac{-2x+3}{3x^2+1}$

$\lim_{x\to\infty} \dfrac{-2x+3}{3x^2+1} = \boxed{0}$

HA: $y=0$

b. $g(x)=\dfrac{-2x^3+3}{3x^2+1}$

$\lim_{x\to\infty} \dfrac{-2x^3+3}{3x^2+1} = \boxed{DNE}$

SA

c. $h(x)=\dfrac{-2x^2+3}{3x^2+1}$

$\lim_{x\to\infty} \dfrac{-2x^2+3}{3x^2+1} = \boxed{-2/3}$

HA: $y=-2/3$

6. You are manufacturing a product that costs \$0.50 per unit to produce. Your initial investment is \$5000, which implies that the total cost of producing x units is C = 0.5x +5000. The **average cost per unit** is given by $\overline{C}=\dfrac{C}{x}=\dfrac{0.5x+5000}{x}$. Find the average cost per unit when

a. x = 1000

$\bar{C} = \dfrac{.5(1000)+5000}{1000}$

$=\boxed{5.5}$

b. x = 10,000

$\bar{C} = \dfrac{.5(10,000)+5000}{10,000}$

$=\boxed{1}$

c. x = 100,000

$\bar{C} = \dfrac{.5(100,000)+5000}{100,000}$

$=\boxed{.55}$

d. What is the limit of $\overline{C}$ when x approaches infinity?

$\lim_{x\to\infty} \bar{C} = \boxed{.5}$

HA: $y=.5$

50¢ per pencil

Day 4: Limit Definition of a Derivative

Tangent Line to a Graph and Slope of a Graph

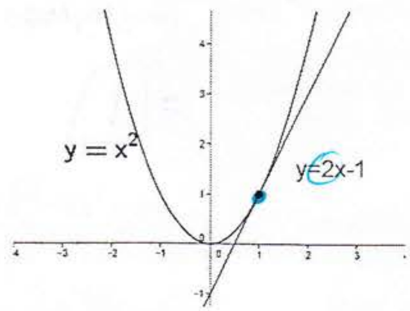
To find the rate at which a graph rises or falls (increases or decreases) at a single point, you can find the slope of the tangent line at that point. This slope represents the **instantaneous rate of change** (IROC) of y with respect to x $\left(\frac{dy}{dx}\right)$ at that point.

We can find IROC by using $\frac{dy}{dx}$ from the calc menu on your calculator.

In simple terms, the tangent line to a graph at a point is a line which best approximates the graph at that point. The slope of the tangent line at a point on a graph basically represents the "slope" of the graph at that point.

1. Use the graph below to approximate the slope of the graph of $y = x^2$ at the point $(1, 1)$.

find slope of tangent line @ $x=1$

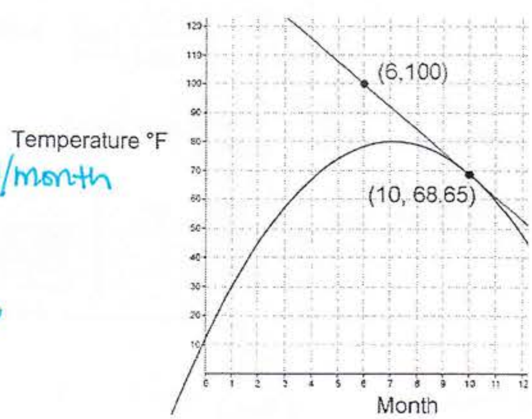


$\boxed{m=2}$

2. The graph below represents the average daily temperature (in degrees Fahrenheit) for each month in Dallas, Texas. Estimate the slope of this graph at the indicated point and give a physical interpretation of the result.

$$m = \frac{100-68.65}{6-10} = \frac{31.35}{-4}$$

$\boxed{m = -7.8375}$ deg/month



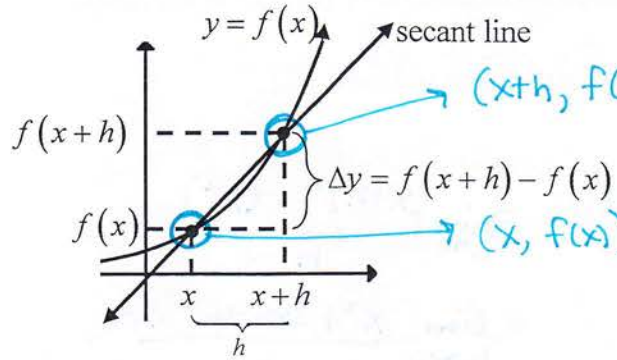
* You can expect the average monthly temperature in November to be ≈ 8° lower than the average temp in October.

Hard to draw tangent line accurately ourselves.

Slope and Limit Process

Visual approximations provide an imprecise way of finding slopes (rates of change). A more precise method involves secant lines and the limit process.

In the figure below, suppose you wish to find the "slope" of the graph of y = f(x) at the point $(x, f(x))$. Using a secant line passing through that point and another point h units to the right (as shown) you can find the slope of the secant line to be:



$(x+h, f(x+h))$

$(x, f(x))$

$$\text{Slope} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$

Average Rate of Change = Slope of secant line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

AROC

$$\boxed{m_{\sec} = \frac{f(x+h) - f(x)}{h}}$$

This is a slope formula. The closer the points are, the better the approximation of the slope of the tangent line.

The slope of a graph = the slope of a tangent line and is given by:

$$m = \lim_{h \to 0} m_{\sec} = \boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$, provided that this limit exists.

derivative

IROC

h = distance between points

as points get closer and closer $h \to 0$

This represents the instantaneous rate of change at $(x, f(x))$.

Derivative of a Function

When you derive the equation for the slopes of a graph, you are finding the derivative of f at x denoted as $\frac{dy}{dx}$ or $f'(x)$.

notation for derivative – they mean the same thing.

Thus $$\boxed{\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

3. Use the definition to find the slope of the graph of $f(x)=x^2$ at any point.

(That is find a general equation for the slope and call it $\frac{dy}{dx}$).

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x^2)}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} 2x+h$$

$$= 2x+0$$

$$= \boxed{2x}$$

this is the formula for the slope at any point on the graph of x^2. (Slope of any line tangent to x^2)

4. Use your answer from #3 to find the slope of the graph of $f(x)=x^2$ at the points:

a. (-2, 4)

Slope when $x=-2$

$$\frac{dy}{dx} = 2x$$

$$= 2(-2)$$

$$= \boxed{-4}$$

b. (0, 0)

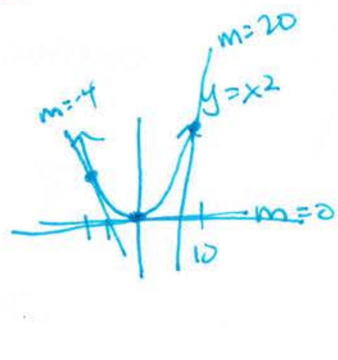
$x=0$

$$\frac{dy}{dx} = 2(0) = \boxed{0}$$

c. (10, 100)

$x=10$

$$\frac{dy}{dx} = 2(10) = \boxed{20}$$



5. Find the derivative of $f(x)=3x^2-2x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{3(x+h)^2-2(x+h)-(3x^2-2x)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{3x^2}+6xh+3h^2-\cancel{2x}-2h-\cancel{3x^2}+\cancel{2x}}{h}$$

$$=\lim_{h\to 0}\frac{6xh+3h^2-2h}{h}$$

$$=\lim_{h\to 0}6x+3h-2$$

$$=6x+3(0)-2$$

$$=\boxed{6x-2}$$

Evaluate the derivative at x = -1 and x = 2, and verify with your calculator.

$f'(-1)=6(-1)-2=\boxed{-8}$ ← Slope of tangent to graph when $x=-1$

$f'(2)=6(2)-2=\boxed{10}$ ← slope of tangent to graph when $x=2$

Write the equation for the tangent line at x = 2. $y-y_1=m(x-x_1)$

* to write equation you need point and slope.

Slope (use $\frac{dy}{dx}$)

$f'(2)=6(2)-2=10$

$m=10$

Point (use f(x) to find y)

$f(2)=3(2)^2-2(2)$

$=3(4)-4$

$=12-4=8$

point: (2,8)

Equation of tangent

$\boxed{y-8=10(x-2)}$

6. Find the derivative of $y = \sqrt{x+2}$. Evaluate the derivative at x = 3 and x = -2, and verify with your calculator.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}\right) \cdot \left(\frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}\right)$$

$$= \lim_{h \to 0} \frac{x+h+2 - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h \to 0} \frac{x+h+2-x-2}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}$$

$$= \frac{1}{\sqrt{x+2+0} + \sqrt{x+2}}$$

$$= \boxed{\frac{1}{2\sqrt{x+2}}}$$

$$f'(3) = \boxed{\frac{1}{2\sqrt{5}}}$$

$$f'(-2) = \frac{1}{2\sqrt{-2+2}} = \frac{1}{0} = \boxed{\text{undefined}}$$

tangent line is vertical @ this point.