Unit 6 Notes / Secondary 3 Honors

Day 1: Exponential Functions

Allison and Beth each receive $10 per week for doing chores for their neighbor. One day Allison decides to try and increase her income using her knowledge of exponential growth. She proposes that her payment be changed to $2 for the first week and then doubled each week thereafter.

1. Complete the table to represent the amount Allison and Beth will earn for the next 8 weeks.

Week	1	2	3	4	5	6	7	8
Allison's $	2	4	8	16	32	64	128	256
Beth's $	10	10	10	10	10	10	10	10

2. Write an equation that will represent Allison's wages for week x. Do the same for Beth's income.

Allison
$y = 2^x$

Beth
$y = 10$

3. How many weeks will it take for Allison's weekly income to surpass Beth's?

4 weeks

Exponential Functions:

variable in exponent

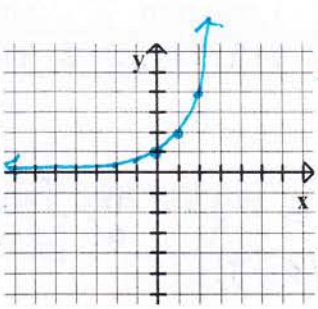
The exponential function with base a (a > 0 and a ≠ 1) is defined by $f(x) = a^x$, where x is any real number.

Exponential Functions	NOT Exponential Functions
$y = 3^x$ $\quad y = 5^{x-2}$ $\quad y = \left(\frac{1}{2}\right)^x$ * variable in exponent	$y = 3^5$, $\quad f(x) = x^2$ $\quad g(x) = x^{-3}$ $h(x) = \dfrac{1}{x^2+1}$

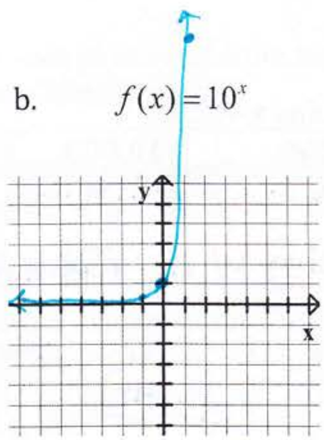
4. Graph the following functions:

a. $y = 2^x$

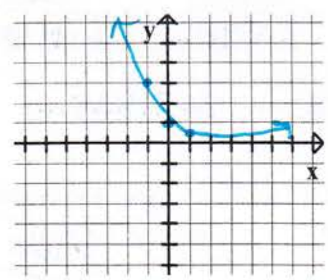
x	y
0	1
1	2
2	4
-1	1/2
-2	1/4



b. $f(x) = 10^x$



c. $y = \left(\frac{1}{3}\right)^x$



5. From your graphs above, try to list some common characteristics for all exponential functions of the form $f(x) = a^x$ (a > 0) and a ≠ 1.

Asymptote: $y = 0$ (x-axis)

y – int: (0,1)

"2nd Point": (1, base)

Domain: ℝ

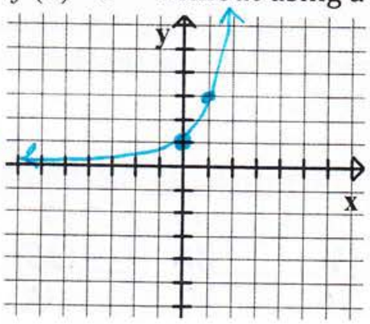
Range: $y > 0$

* All exp. graphs start w/
(0,1)
(1,b)
y=0 asymp

Transformations

6. Graph $f(x)=3^x$, and then describe how you could obtain each of the following graphs from $f(x)=3^x$ without using a calculator.

(0,1)
(1,3)
y=0 asymp



a. $g(x)=3^{x+1}$

left 1

b. $h(x)=3^x-2$

↓2

c. $k(x)=-3^x$

reflect over x-axis

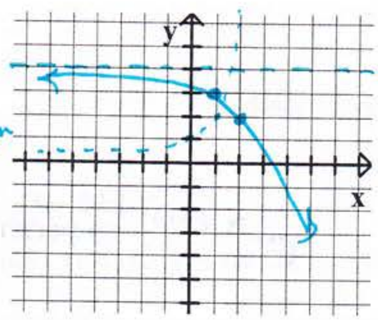
d. $j(x)=3^{-x}$

reflect over y-axis

7. Graph the following functions:

a. $f(x)=-2^{x-1}+4$

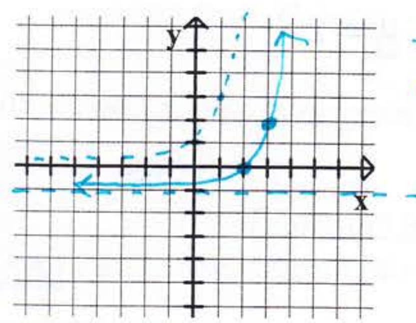
Start
(0,1)
(1,2)
y=0 asym



• reflect over x-axis
• → 1
• ↑ 4

b. $g(x)=3^{x-2}-1$

Start
(0,1)
(1,3)
x-axis asymp



→ 2
↓ 1

The Natural Base e

A very common base for exponential functions is e, where e ≈ 2.718. (It is an irrational number). which means it keeps going & going … $f(x)=e^x$ is known as the natural exponential function. e is by far the most commonly used base in calculus. It is defined as $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

Evaluate this expression for the following x-values:

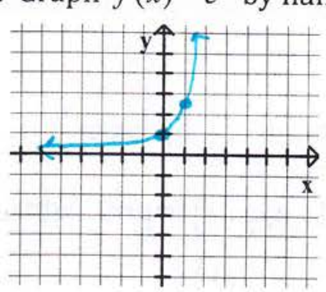
10	100	1,000	10,000	100,000	1,000,000	10^{10}
2.593...	2.7048...	2.716...	2.7181...	2.71827...	2.71828...	2.71828...

So, e = 2.71828.... keeps going w/ no repeating pattern

You can use your calculator to evaluate $f(x)=e^x$ for any given value of x. e key is 2nd function of $\boxed{\ln}$

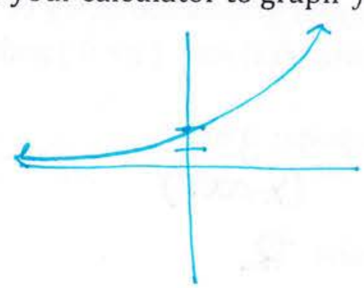
8. a. e^{-2} $\boxed{.135}$ b. e^1 $\boxed{2.718}$ c. e^{π} $\boxed{23.141}$

9. Graph $f(x)=e^x$ by hand:



(0,1)
(1,e)
x-axis asymp.

10. Use your calculator to graph $f(x)=2e^{.24x}$



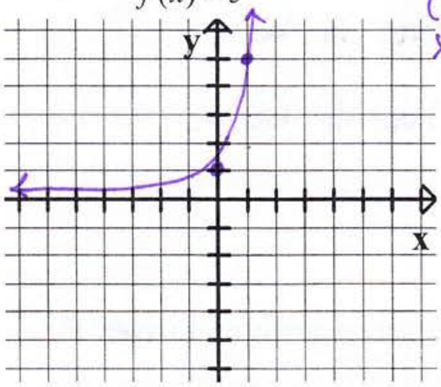
y 2: Logarithmic Functions

*reflect over y=x.
Switch x&y

As a quick review, graph:

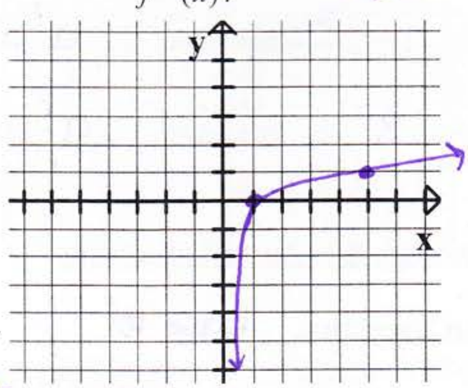
$f(x) = 5^x$

(0,1)
(1,5)
x-axis asymp.



$f^{-1}(x)$.

(1,0)
(5,1)
y-axis asymp.



D: ℝ **R:** y>0 ← D&R switch → **D:** x>0 **R:** ℝ

Every mathematical function has an inverse. An inverse "undoes" the function. For instance, the inverse of multiplication is division. To "undo" a square, it must be square rooted. This is how we solve equations to isolate the variable *x*. * So if we want to solve an exponential equation we need to know its inverse.

The inverse of an exponential function $f(x) = a^x$ is called the logarithmic function with base a.

For $x > 0$ and $a > 0$, $a \neq 1$, $\boxed{y = \log_a x} \leftrightarrow \boxed{x = a^y}$

Rules for bases of exp. functions.

↳ "log base a of x"

- The base of a logarithmic function must be greater than 0 and not equal to 1.
- Also, you cannot take the logarithm of a number which is not positive.

You should be able to use the definition above to convert from logarithmic form to exponential form or vice versa for equations expressed in one of these forms.

1. Rewrite each logarithmic equation in exponential form:
   a. $\log_4 256 = 4$ b. $\log_6 36 = 2$ c. $\log_4 1 = 0$
   $4^4 = 256$ $6^2 = 36$ $4^0 = 1$

2. Rewrite each exponential equation in logarithmic form:
   a. $4^3 = 64$ b. $16^{\frac{1}{2}} = 4$ c. $3^{-1} = \frac{1}{3}$
   $\log_4 64 = 3$ $\log_{16} 4 = 1/2$ $\log_3 \frac{1}{3} = -1$

- Remember that the value of a logarithm is an exponent.
- $\log_a x$ is the exponent to which "a" must be raised to obtain x.

3. Compute each of the following by converting them to exponential form: (non-calculator)
   * trying to find the exponent
   a. $\log_2 8 = \boxed{3}$ b. $\log_2 2 = \boxed{1}$ c. $\log_3 1 = \boxed{0}$ d. $\log_5 \frac{1}{25} = \boxed{-2}$ e. $\log_9 3 = \boxed{1/2}$
   $2^? = 8$ $2^? = 2$ $3^? = 1$ $5^? = \frac{1}{25}$ $9^? = 3$

Basic Properties of Logarithms:

$\log_a 1 =$ ___0___ because: $a^0 = 1$

$\log_a a =$ ___1___ because: $a^? = a$ power has to be 1

$\log_a a^x =$ ___x___ because: $a^? = a^x$ power has to be x

Common Logarithm & Natural Logarithm

Common Logarithm base 10

$f(x) = \log(x)$ (same as $\log_{10} x$)

Natural Logarithm base e

$f(x) = \ln(x)$ (means $\log_e x$ but always use ln notation)

4. Use your calculator to evaluate

a. $\log 100 = \boxed{2}$ b. $\log 12\ \boxed{1.079}$ c. $\log(-2) = \boxed{\text{undefined}}$

why? $10^? = -2$ (can't raise a + number to a power and get a - answer)

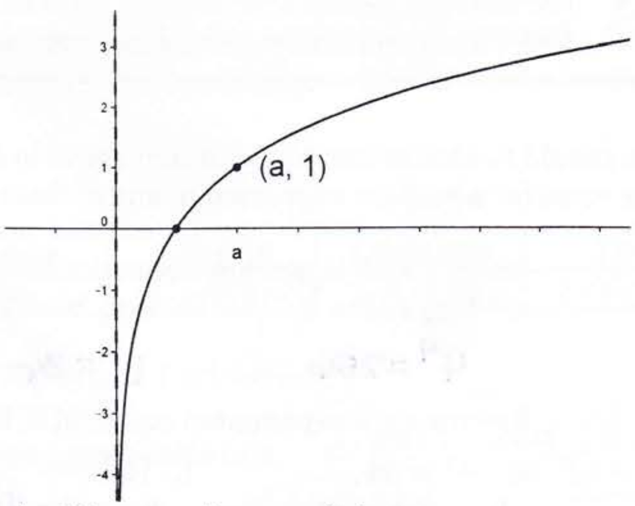
d. $\log e^8\ \boxed{3.474}$ e. $-2\ln 5\ \boxed{-3.219}$ f. $\ln \frac{1}{4}\ \boxed{-1.386}$

Graphs of Logarithmic Functions

Every exponential function $y = a^x$ passes through the points (0, 1) and (1, a) with $y = 0$ for a horizontal asymptote.

Every logarithm function $y = \log_a(x)$ passes through the points (1, 0) and (a, 1) with $x = 0$ for a vertical asymptote.

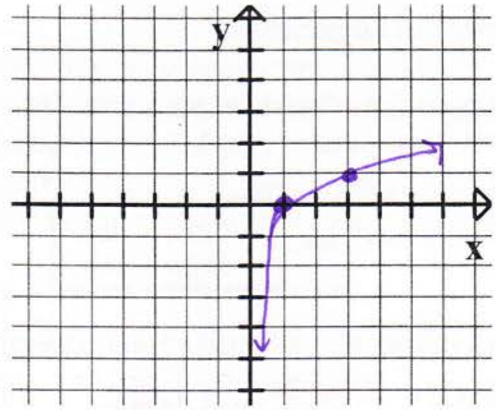
* just use inverse properties to graph logs :)
*switch x & y



In general, the graph of $f(x) = \log_a x$ where $a > 1$ has the following characteristics:

- D: x > 0, R: all reals
- x-int: (1, 0) and another key point (a , 1)
- VA: y axis

Graph $y = \log_3 x$



D: $x > 0$

R: $\mathbb{R}$

Key points: $(1,0)$ $(3,1)$

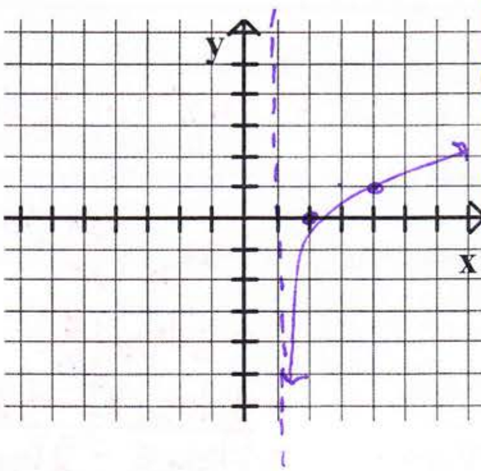
Asymptote: y-axis

6. Use your knowledge of basic logarithmic graphs and transformations to graph the following:

a. $f(x) = \log_3(x-1)$

Start
(1,0) (3,1) y-axis

→ 1



D: $x > 1$

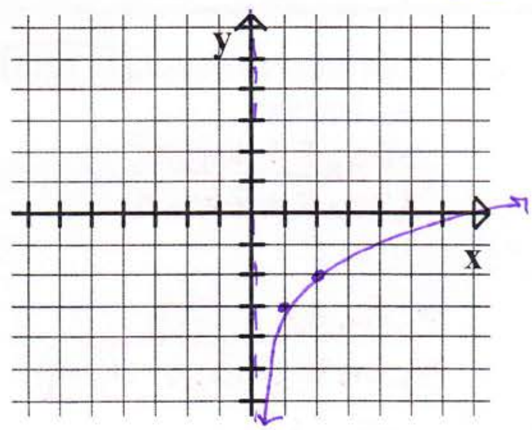
R: $\mathbb{R}$

A: $x = 1$

b. $p(x) = -3 + \log_2 x$

Start
(1,0) (2,1)
y-axis

↓ 3



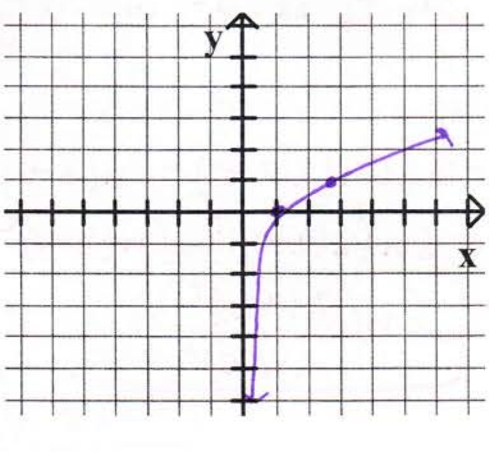
D: $x > 0$

R: $\mathbb{R}$

A: $x = 0$

c. $q(x) = \ln x$
↓
base is e

Start
(1,0)
(e,1) y-axis



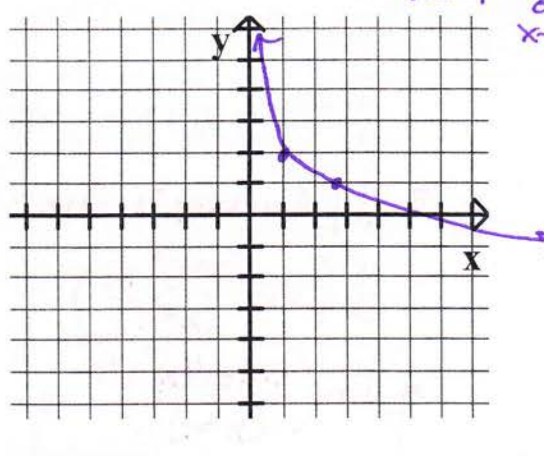
D: $x > 0$

R: $\mathbb{R}$

A: $x = 0$

d. $g(x) = 2 - \ln x$

Start
(1,0) (e,1)
y-axis
↑2 & reflect over x-axis



D: $x > 0$

R: $\mathbb{R}$

A: $x = 0$

Day 3: Properties of Logarithms

Since the only logarithmic bases on your calculator are 10 (log key) and e (ln key), you need to know how to change bases to compute or graph logarithmic expressions or functions.

Change of Base Formula: $\boxed{\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}}$

* Newer Calculators have this in Shortcut menu (Alpha F2)

1. Find each of the following using your calculator.

a. $\log_4 30 = \frac{\log 30}{\log 4} = \boxed{2.453}$

b. $\log_2 14 = \frac{\log 14}{\log 2} = \boxed{3.807}$

c. $\log_6 \frac{2}{3} = \frac{\log \frac{2}{3}}{\log 6} = \boxed{-.226}$

Properties of Logarithms

- $\log_a xy = \log_a x + \log_a y$ → power property
- $\log_a \frac{x}{y} = \log_a x - \log_a y$ → Quotient property
- $\log_a x^y = y \log_a x$ → "power to power"

2. Use the properties to expand the expressions.

a. $\log_3 5x^2$

$= \log_3 5 + \log_3 x^2$

$= \boxed{\log_3 5 + 2\log_3 x}$

b. $\ln\sqrt{x} = \ln x^{1/2} = \boxed{\frac{1}{2}\ln x}$

c. $\ln\frac{3x}{y}$

$= \ln 3x - \ln y$

$= \boxed{\ln 3 + \ln x - \ln y}$

d. $\log_{10}\sqrt[3]{\frac{z^3}{x^2}} = \log_{10}\left(\frac{z^3}{x^2}\right)^{1/3}$

$= \frac{1}{3}\log_{10}\left(\frac{z^3}{x^2}\right)$

$= \frac{1}{3}\log_{10} z^3 - \frac{1}{3}\log_{10} x^2$

$= \boxed{\log_{10} z - \frac{2}{3}\log_{10} x}$

3. Use the properties to condense the expression to a logarithm of a single quantity.

a. $\log_4 8 - \log_4 t$

$= \boxed{\log_4\left(\frac{8}{t}\right)}$

b. $\ln x - 3\ln y + 2\ln z$

$= \ln x - \ln y^3 + \ln z^2$

$= \ln\frac{x}{y^3} + \ln z^2 = \boxed{\ln\left(\frac{xz^2}{y^3}\right)}$

c. $\frac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)]$

$= \frac{1}{3}\left[\ln(x+3)^2 + \ln x - \ln(x^2-1)\right]$

$= \frac{1}{3}\left(\ln\frac{(x+3)^2(x)}{(x^2-1)}\right)$

$= \boxed{\ln\left(\frac{(x+3)^2(x)}{(x^2-1)}\right)^{1/3}}$

d. $4[\ln z + \ln(z+5)] - 2\ln(z-5)$

$= 4(\ln(z(z+5)) - \ln(z-5)^2$

$= \ln(z(z+5))^4 - \ln(z-5)^2$

$= \boxed{\ln\left(\frac{(z(z+5))^4}{(z-5)^2}\right)}$

…n't forget:

- When expanding the logarithm, use parentheses to clarify your work if needed.
- When condensing the expression, work from left to right, but if there is parentheses use order of operation thinking to put the expression back together.
- If unsure if the expression is the same, substitute numbers for the variables and check both the expanded and condensed logarithm with your calculator to see if they are the same.

5. Write each logarithm in terms of: $\begin{cases} \log_x 2 = .9 \\ \log_x 3 = 1.2 \end{cases}$

a. $\log_x 6$

$= \log_x 2 \cdot 3$

$= \log_x 2 + \log_x 3$

$= .9 + 1.2 = \boxed{10.2}$

b. $\log_x 72$

$\log_x 2^3 \cdot 3^2$

$= 3\log_x 2 + 2\log_x 3$

$= 3(.9) + 2(1.2)$

$= \boxed{5.1}$

c. $\log_x \frac{2}{27} = \log_x \frac{2}{3^3}$

$= \log_x 2 - 3\log_x 3$

$= .9 - 3(1.2)$

$= \boxed{-2.7}$

<u>Day 4</u>: Solving Exponential and Logarithmic Equations

<u>Solving Exponential Equations</u>

To solve an exponential equation, isolate the exponential expression and switch forms.

Solve algebraically. Round answers to 3 decimal places. * Wait to use calc until end of problem.

1. $8^{3x} = 360$

$\log_8 360 = 3x$

$\frac{\log_8 360}{3} = x$

$x = \boxed{.944}$

2. $e^x = 72$

$\ln 72 = x$

$\boxed{4.277 = x}$

3. $5 \cdot 2^{3x-1} = 15$

$2^{3x-1} = 3$

$\log_2 3 = 3x - 1$ $\boxed{x = .862}$

$\frac{\log_2(3) + 1}{3} = x$

4. $3e^{2x} + 5 = 18$

$3e^{2x} = 13$

$e^{2x} = 13/3$ $\boxed{x = .733}$

$\ln 13/3 = 2x$

$\frac{\ln 13/3}{2} = x$

5. $6(8^{-2-x}) + 15 = 2601$

$6(8^{-2-x}) = 2586$

$8^{-2-x} = 431$

$\log_8 431 = -2 - x$

$\log_8 431 + 2 = -x$

$x = -\log_8 431 - 2$

$\boxed{x = -4.917}$

6. $e^x = e^{x^2-2}$ if bases are same then exponents must be =

$x = x^2 - 2$

$0 = x^2 - x - 2$

$= (x-2)(x+1)$

$\boxed{x = 2,\ x = -1}$

7. $e^{2x} - 3e^x + 2 = 0$ (hint: factor first)

$(e^x - 2)(e^x - 1) = 0$

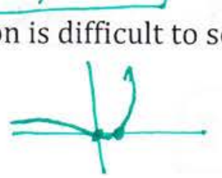
$e^x - 2 = 0$ $e^x - 1 = 0$

$e^x = 2$ $e^x = 1$

$\ln 2 = x$ $\ln 1 = x$

$\boxed{.693 = x}$ $\boxed{0 = x}$

If an equation is difficult to solve algebraically, solve graphically. Solve #7 above graphically.



$\boxed{x = 0,\ .693}$

Solving Logarithmic Equations

To solve a logarithmic equation, isolate a logarithm and switch forms. Check answers for extraneous solutions. (Remember you can't have 0 or a negative number inside a log.)

Solve algebraically. Round answers to 3 decimal places.

8. $\log_{10}(z-3) = 2$

$10^2 = z-3$

$10^2 - 3 = z$

$\boxed{97 = z}$

9. $\ln(x^2+1) = 8$

$e^8 = x^2+1$

$e^8 - 1 = x^2$

$\pm\sqrt{e^8-1} = x$

$\boxed{x = \pm 54.589}$

10. $5 + 2\ln x = 4$

$2\ln x = -1$

$\ln x = \frac{-1}{2}$

$e^{-1/2} = x$

$\boxed{x = .607}$

11. $\log(x) - \log(x-2) = 1$

$\log \frac{x}{x-2} = 1$

$10^1 = \frac{x}{x-2}$

$10x - 20 = x$

$9x = 20$

$x = 20/9 = \boxed{2.\overline{2}}$

12. $\ln(x-2) + \ln(2x-3) = 2\ln x$

(If $\ln x = \ln y$ then $x = y$)

$\ln((x-2)(2x-3)) = \ln x^2$

$\ln(2x^2 - 7x + 6) = \ln x^2$

$2x^2 - 7x + 6 = x^2$

$x^2 - 7x + 6 = 0$

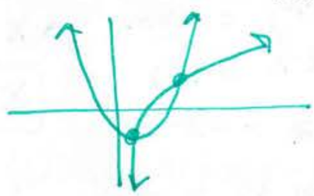
$(x-6)(x-1) = 0$

$\boxed{x = 6}$

$\cancel{x = 1}$

Difficult equations and/or equations involving both exponential and logarithmic expressions may be difficult (or even impossible) to solve algebraically. Approximate solutions can be found graphically.

13. Approximate the solution(s) of $2\ln x = x^2 - 2$ graphically.



$\boxed{\begin{matrix} x = .398 \\ x = 1.773 \end{matrix}}$

14. From 1990 to 2013, the Consumer Price Index (CPI) value y for a fixed amount of sugar for the year t can be modeled by the equation y = -169.8 + 86.9 ln t, where t = 10 represents 1990. During which year did the price of sugar reach 4 times its 1990 price of 30.5 on the CPI?

$4(30.5) = 122$

$122 = -169.8 + 86.9\ln t$

$291.8 = 86.9\ln t$

$\frac{291.8}{86.9} = \ln t$

$t = e^{291.8/86.9} = 28.73 \rightarrow$ 28 years past 1980

$\boxed{\text{Year} = 2008}$

ay 5: Exponential and Logarithmic Applications

Logarithmic Applications:

Measurements of earthquake magnitudes, sound intensity, and pH all use logarithmic models.

1. On the Richter scale, the magnitude R of an earthquake of intensity I is given by: $R = \log_{10}\frac{I}{I_0}$ where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensities per unit area for the following earthquakes. (Intensity is a measure of the wave energy of the earthquake.)

   a. Tokyo and Yokohama, Japan, in 1923, R = 8.3

   $R = \log_{10} I$

   $8.3 = \log_{10} I$

   $10^{8.3} = I$

   $\boxed{I = 199{,}526{,}231.5}$

   b. Kobe, Japan, in 1995, R = 7.2

   $7.2 = \log_{10} I$

   $10^{7.2} = I$

   $\boxed{I = 15{,}848{,}931.92}$

Exponential Models

Compound Interest

Most financial institutions allow interest to compound- that is, they pay interest on interest. Suppose P dollars are invested at rate r with interest compounded n times each year. Then the amount available at the end of t years is:

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

P = principal (initial amt)

n = # of compounding periods / year

r = annual interest rate (decimal form)

t = time (years)

2. Suppose you invest \$500 and the money was compounded yearly (annually). How much money would be in your account now at the end of 10 years if the interest rate is 2%?

   $A = 500\left(1+\frac{.02}{1}\right)^{1(10)} = 500(1+.02)^{10} = \boxed{\$609.50}$

3. Suppose your original investment of \$500, at 2% interest per year, is compounded: a. quarterly b. monthly. How much money would be in your account after 10 years?

   a.

   $A = 500\left(1+\frac{.02}{4}\right)^{4(10)}$

   $= \boxed{\$610.40}$

   b.

   $A = 500\left(1+\frac{.02}{12}\right)^{12(10)}$

   $= \boxed{\$610.60}$

Continuously Compounded Interest:

$\boxed{A = Pe^{rt}}$ Interest is compounded an infinite number of times per year.

4. If you invest $5000, how long will it take for the amount to double if it is invested at 9.5% compounded continuously?

$10{,}000 = 5000\,e^{.095t}$

$2 = e^{.095t}$

$\ln 2 = .095t$

$t = \frac{\ln 2}{.095} = \boxed{7.3 \text{ years}}$

Exponential Growth or Decay:

Population growth, compound interest, and half-life problems are exponential growth and decay models. Any exponential growth or decay problem can be expressed with the formula:

$\boxed{A = Ce^{kt}}$ growth: k > 0 and decay: k < 0.

Typo 5. The sales S (in thousands of units) of a cleaning solution after X hundred dollars is spent on advertising are $S = 10(1 - e^{kx})$. You know that when $500 is spent on advertising, 2500 units are sold. $x = 5$ $S = 2.5$

a. Complete the model by solving for k.

$2.5 = 10(1 - e^{k(5)})$

$.25 = 1 - e^{5k}$

$-.75 = -e^{5k}$

$.75 = e^{5k}$

$\ln .75 = 5k$

$\frac{\ln(.75)}{5} = k$

$\approx -.0575$

b. Estimate the number of units that will be sold if advertising expenditures are raised to $700.

$S = 10(1 - e^{-.0575(7)}) = 3.314$ thousand

$= \boxed{3{,}314}$

6. Find an equation for exponential growth for a colony of bacteria if the initial amount of bacteria is 50 and after 6 hours the amount of bacteria is 1280.

(t, bacteria)

(0, 50) – use initial for C.

(6, 1280) – use this to find k.

Exp growth

$A = Ce^{kt}$

$A = 50e^{kt}$

$1280 = 50e^{k(6)}$

$25.6 = e^{6k}$

$\ln 25.6 = 6k$

$\frac{\ln 25.6}{6} = k$

$.5404 = k$

Should be "+" why?

$\boxed{A = 50e^{.5404t}}$

The **half-life** of radium is 1690 years.

half-life = amount of time for initial amount to be cut in half.

a. Write an equation representing the situation. Round k to at least 4 decimal places.

decay (so k is "–") if half-life $\frac{A}{C} = \frac{1}{2}$

$A = Ce^{kt}$

$\frac{A}{C} = e^{k(t)}$

$\frac{1}{2} = e^{k(1690)}$

$\ln\frac{1}{2} = 1690k$

$k = \frac{\ln 1/2}{1690} \approx -.00041$

$\boxed{A = Ce^{-.00041t}}$

b. If 10 grams are present now, how much will be present in 50 years?

$A = 10e^{-.00041(50)}$

$\boxed{A = 9.797\,g}$