Unit 10
Logarithmic and Exponential Functions

Day 1
Properties of Logarithms

For any base other than 10 or e, you must use the **Change of Base Formula** to evaluate logarithms with a calculator. (Some calculators include a change-of-base formula in their internal functions, but you should know the change of base formula because not all calculators do.)

To change from base a to base 10 or base e, use the following formula:

To base e: $\log_a N = \dfrac{\ln N}{\ln a}$

To base 10: $\log_a N = \dfrac{\log N}{\log a}$

You can use either formula; just remember: the "old" base goes in the "basement".

Examples. Use either form of the Change of Base Formula to evaluate:

1. $\log_4 30 = \dfrac{\log 30}{\log 4} = 2.453445$
2. $\log_2 0.015 = \dfrac{\log 0.015}{\log 2} = -6.05889$

The following **Basic Properties of Logarithms** follow directly from the definition: $y = \log_a x \leftrightarrow x = a^y$
➤ $\log_a 1 = 0$, because $a^0 = 1$
➤ $\log_a a = 1$, because $a^1 = a$
➤ $\log_a a^n = n$ and $a^{\log_a n} = n$ **(Inverse properties)**
➤ If $\log_a x = \log_a y$, then $x = y$ **(one-to-one property)**

Examples.

3. Use the basic properties of logarithms to simplify the expression.

a. $\cancel{\log_3 3^x}$ $3^? = 3^x$ $\boxed{x}$ or $\boxed{x}$

b. $\cancel{5^{\log_5}}7$ $\boxed{7}$

c. $\cancel{\ln e}^{2x}$ $e^? = e^{2x}$ $\boxed{2x}$ or $\boxed{2x}$

d. $\cancel{\log 10}$ $10^? = 10$ $\boxed{1}$ or $\boxed{1}$

4. Use the basic properties of logarithms to solve for x.

a. $\cancel{\log_6} x = \cancel{\log_6} 17$

$x = 17$

b. $\log_{17} 17 = x$

$17^x = 17$

$x = 1$

Other Properties of Logarithms (Expanding and Condensing)

The following major properties of logarithms can be developed from corresponding properties of exponents. They work for **positive values of x and y**.

Properties of Logarithms
- **Product property**: $\log_a(mn) = \log_a m + \log_a n$
- **Quotient property**: $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
- **Power property**: $\log_a m^n = n\log_a m$

When using the power property with square and cube roots in the argument, always rewrite the term using rational exponents instead of roots: $\sqrt[n]{x} = x^{1/n}$

You can use the properties of logarithms to evaluate logarithms, even if you don't know the base.

Examples. Given $\log_a 2 \approx 0.431$, $\log_a 3 \approx 0.682$. and $\log_a 7 \approx 1.209$, use the properties of logarithms to evaluate:

5. $\log_a 6$

$= \log_a(2\cdot 3)$

$= \log_a 2 + \log_a 3$

$= 0.431 + 0.682$

$= \boxed{1.113}$

6. $\log_a\left(\frac{7}{3}\right)$

$= \log_a 7 - \log_a 3$

$= 1.209 - 0.682$

$= \boxed{0.527}$

7. $\log_a 8$

$= \log_a 2^3$

$= 3\log_a 2$

$= 3(0.431)$

$= \boxed{1.293}$

8. $\log_a 63$

$= \log_a(9\cdot 7)$

$= \log_a 9 + \log_a 7$

$= \log_a 3^2 + \log_a 7$

$= 2\log_a 3 + \log_a 7$

$= 2(0.682) + (1.209)$

$= \boxed{2.573}$

Examples. Use the properties of logarithms to rewrite each expression in terms of ln5:

9. $\ln 25$

$= \ln(5\cdot 5)$

$= \ln 5 + \ln 5$

$= \boxed{2\ln 5}$

-or-

$= \ln 5^2$

$= \boxed{2\ln 5}$

10. $\ln\left(\frac{1}{5}\right)$

$= \ln 1 - \ln 5$

$= 0 - \ln 5$

$= \boxed{-\ln 5}$

-or-

$= \ln 5^{-1}$

$= \boxed{-\ln 5}$

11. $\ln 125 - 4\ln 5$

$= \ln 5^3 - 4\ln 5$

$= 3\ln 5 - 4\ln 5$

$= \boxed{-\ln 5}$

Assignment 10.1

Day 2

Expanding and Condensing Logarithmic Expressions

The properties of logarithms can be used to rewrite logarithmic expressions that include variables.

Examples.
Use the properties of logarithms to **expand** each expression (**rewrite as a sum or difference of logarithms**).

1. $\log_2(2xy)$

$= \log_2 2 + \log_2 x + \log_2 y$

$= \boxed{1 + \log_2 x + \log_2 y}$

2. $\log_3\left(\frac{4x}{z}\right)$

$= \log_3 4x - \log_3 z$

$= \boxed{\log_3 4 + \log_3 x - \log_3 z}$

3. $\log(x^3\sqrt{y})$

$= \log x^3 + \log \sqrt{y}$

$= 3\log x + \log y^{1/2}$

$= \boxed{3\log x + \frac{1}{2}\log y}$

4. $\ln\left(\frac{y^3}{xz^2}\right)$

$= \ln y^3 - \ln x - \ln z^2$

$= \boxed{3\ln y - \ln x - 2\ln z}$

*anything in the denominator will have a negative in front of it.

Examples. Use the properties of logarithms to **condense** each expression (**write as a single logarithm**).

5. $\frac{1}{3}\log x + 5\log(x-3)$

$= \log x^{1/3} + \log(x-3)^5$

$= \log \sqrt[3]{x} + \log(x-3)^5$

$= \boxed{\log \sqrt[3]{x}(x-3)^5}$

6. $4\ln(x-4) - 2\ln x$

$= \ln(x-4)^4 - \ln x^2$

$= \boxed{\ln\left(\frac{(x-4)^4}{x^2}\right)}$

7. $\frac{1}{5}[\log_3 x + \log_3(x-2)]$

$= \frac{1}{5}[\log_3 x(x-2)]$

$= \log_3 x(x-2)^{\frac{1}{5}}$

or

$\log_3 \sqrt[5]{x(x-2}$

8. Find the mistake in each of these equations and rewrite as a true statement.

a. $\log(x+y) = \log x + \log y$ $\quad\quad$ $\log(xy) = \log x + \log y$

b. $\log_3\left(\frac{x}{y}\right) = \frac{\log_3 x}{\log_3 y}$ $\quad\quad$ $\log_3\left(\frac{x}{y}\right) = \log_3 x - \log_3 y$

c. $2\ln xy = \ln xy^2$ $\quad\quad$ $2\ln xy = \ln(xy)^2$

Assignment 10.2

Day 3

Solving Exponential Equations

An <u>exponential equation</u> is an equation where the variable is in the exponent. We will explore two ways to solve exponential equations.

Solve by Writing in Logarithmic Form

- Isolate the exponential expression, using addition, subtraction, multiplication or division on both sides of the equation.
- Write the equation in logarithmic form.
- Solve for the variable.
- You may need to use a change-of-base formula to obtain an approximate value of the solution.

Examples. Solve the following exponential equations:

1. $5^x = 10$

$\log_5 10 = x$

↑ type in calc

$\boxed{x = 1.43}$

2. $3e^x + \cancel{11} = 14$

Isolate

-11

$\frac{\cancel{3}e^x}{\cancel{3}} = \frac{3}{3}$

$e^x = 1$ Switch forms → $\ln_e 1 = x$

type in cal

$\boxed{x = 0}$

3. $3\cdot 2^{3x-4} - \cancel{18} = 0$

$+18$

$\frac{\cancel{3}\cdot 2^{3x-4}}{\cancel{3}} = \frac{18}{3}$

$2^{3x-4} = 6$ Switch forms

↓

$\log_2 6 = 3x - 4$

$+4 \quad +4$

$\frac{\log_2 6 + 4}{3} = \frac{\cancel{3}x}{\cancel{3}}$

$\boxed{x = 2.19}$

4. $\frac{30}{e^{2x}-1} = \frac{15}{1}$

$30 = 15e^{2x} - 15$

$+15 \quad +15$

$\frac{45}{15} = \frac{\cancel{15}e^{2x}}{\cancel{15}}$

$3 = e^{2x}$

switch forms

$\frac{\ln_e 3}{2} = \frac{\cancel{2}x}{\cancel{2}}$

$\boxed{x = .55}$

Solve by taking the logarithm of both sides

Some exponential equations involve more than one exponential term. You can take the logarithm of both sides if the equation can first be written as a log equal to a log. Use the properties of logarithms to simplify.

Examples. Solve the following exponential equations.

5. $\cancel{\ln} e^{x^2+6} = \cancel{\ln} e^{5x}$

only 1 e on each side

$x^2 + 6 = 5x$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$\boxed{x = 3 \quad x = 2}$

6. $2^{x-3} - 2^{2x+1} = 0$

rearrange to get 1 on each side.

$\cancel{\log_2} 2^{x-3} = \cancel{\log_2} 2^{2x+1}$

$x - 3 = 2x + 1$

$-x$

$-3 = x + 1$

$-1 \quad -1$

$\boxed{x = -4}$

Assignment 10.3

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Day 4

Solving Logarithmic Equations

A <u>logarithmic equation</u> is an equation where the variable is in the argument of a logarithm. We will explore two ways to solve logarithmic equations.

Solve by Writing in Exponential Form

1. First isolate the logarithmic expression. This may require use of the properties of logarithms.
2. Write the equation in exponential form.
3. Solve for the variable.
4. Check for extraneous solutions (especially if you solved a quadratic to get the answers). Do the solutions work in the original equation?

Examples. Solve the following logarithmic equations for x. Be sure to check solutions:

1. $2\ln(5x) = 12$

$\frac{2\ln(5x)}{2} = \frac{12}{2}$

$\ln(5x) = 6$ Switch forms

$\frac{e^6}{5} = \frac{5x}{5}$

$\boxed{x = 80.69}$

Check ✓

2. $\log(2x-1)^2 = 6$

$\frac{2\log(2x-1)}{2} = \frac{6}{2}$

$\log(2x-1) = 3$

Switch forms

$10^3 = 2x-1$

$1000 + 1 = 2x - 1 + 1$

$\frac{1001}{2} = \frac{2x}{2}$

$\boxed{x = 500.5}$ ✓

3. $\log x + \log(x-9) = 1$

Condense

$\log x(x-9) = 1$

Switch forms

$10^1 = x(x-9)$

$10 = x^2 - 9x$

$0 = x^2 - 9x - 10$

$0 = (x-10)(x+1)$

$\boxed{x = 10}$ ✓ $\cancel{x = -1}$

Gives a negative

4. $\log_4 x - \log_4(x-3) = 1$

Condense

$\log_4 \frac{x}{x-3} = 1$

Switch forms

$\frac{4^1}{1} = \frac{x}{x-3}$

$4(x-3) = x$

$4x - 12 = x$

$-12 = -3x$

$\boxed{x = 4}$ ✓

Solve by Using the One-to-One Property

The one-to-one property of logs states: $\log_a x = \log_a y$ then $x = y$. So if a logarithmic equation can be written in the form $\log_a x = \log_a y$, no matter what the base is, it will be most easily solved by setting the arguments equal to each other.

Examples. Solve the following equations for x.

5. $\log_3(x+2) - \log_3 x^2 = 0$

Get one on each side

$\log_3(x+2) = \log_3 x^2$

$x + 2 = x^2$

$0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$\boxed{x = 2 \quad x = -1}$ ✓

6. $\ln(3x) = \ln(6x-14)$

Only one on each side

$3x - 6x = 6x - 14 - 6x$

$-3x = -14$

$\boxed{x = \frac{14}{3}}$ ✓

Solve Exponential and Logarithmic Equations Using a Graphing Calculator

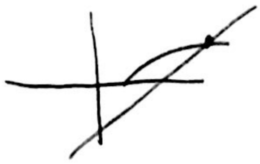
Sometimes an exponential or logarithmic equation cannot be solved algebraically. But approximate solutions can be found by graphing each side of the equation in a graphing calculator and finding the point(s) of intersection. The x value(s) of these points are the solutions to the equation.

Examples. Use a calculator to find solution(s) for the following equations. Show a graph as justification. Round answers to three decimal places.

7. $e^{x^2} - 1 = \sqrt{\ln(x+4)}$

$\boxed{\begin{matrix} -.853 \\ .903 \end{matrix}}$

8. $\log(2x-4) = x-4$



$\boxed{8.085}$

Assignment 10.4

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Day 5

Compound Interest

A common business application of the number e occurs in computing interest for investments.

- **<u>Non-continuous compounding</u>**: Interest is calculated a specific number of times each year. (Daily, weekly, monthly, quarterly, semi-annually, and annually are some common types of compoundings.)
- **<u>Continuous compounding</u>**: Interest is always being calculated - an infinite number of times.

Given on test ↓   need to memorize ↓

Compound Interest: $A = P\left(1+\frac{r}{n}\right)^{nt}$	**A = Amount (final)** **P = Principal (initial amount)**
	r = rate **t = time (yrs)**
Continuous Interest: $A = Pe^{rt}$	**n = number of compounds per year**

Examples.

1. Suppose you have \$10,000 to invest in a college fund, and you can leave the money in the fund for 8 years before you start college. Which of the following would provide you with the most money for college?

   a. A fund providing an interest rate of 2.3% **compounded semi-annually.**

   $A = 10000\left(1 + \frac{.023}{2}\right)^{2(8)}$

   $A = \$12,007.54$

   b. A fund providing an interest rate of 2.3% **compounded continuously.**

   $A = 10000e^{.023(8)}$

   $A = \$12,020.16$ ✱ better

2. How much should you invest (P) in an account with an interest rate of 4% (r) that is compounded quarterly (n) if you want to have \$25,000 (A) in 20 years (t)?

$\$25,000 = P\left(1+\frac{.04}{4}\right)^{4(20)}$

$\frac{25000}{2.2167} = \frac{P(\cancel{2.2167})}{\cancel{2.2167}}$

$\boxed{P = \$11,277.95}$

3. \$500 is deposited into an account that pays 6.75% interest compounded continuously. How long will it take the money to double in value?

$\$1000$ because $\$500(2)$

$\frac{1000}{500} = \frac{500e^{.0675t}}{500}$

$2 = e^{.0675t}$

$\frac{\ln 2}{.0675} = \frac{.0675t}{.0675}$

$\boxed{t = 10.3 \text{ yrs.}}$

Other Applications Involving Exponential Models

Many common applications of exponential growth (or decay) occur in the sciences.

Exponential Growth & Decay: $\boldsymbol{y = Ce^{kt}}$ C = initial amount $\quad$ t = time $\quad$ y = Amount at time t $k > 0$ growth constant , $k < 0$ decay constant

Examples.

4. In Bolivia, the population was 10.5 million people in the year 2010. It's estimated the country has a 1.6% annual growth rate.
   a. Write a function to model Bolivia's population with respect to t, the number of years since 2010. Write the function in the form $y = Ce^{kt}$

   $y = 10.5e^{.016t}$

   b. Use the model to predict Bolivia's population in the year 2030.

   $y = 10.5e^{.016(20)} = \boxed{14.5 \text{ million people}}$

   c. Using this model, what was Bolivia's population in 1990?

   $y = 10.5e^{.016(-20)} = \boxed{7.6 \text{ million people}}$

   d. Using this model, how long will it take Bolivia's population to double?

   $10.5(2) = 21$

   $\frac{21}{10.5} = \frac{10.5e^{.016t}}{10.5}$

   $2 = e^{.016t} \rightarrow \frac{\ln 2}{.016} = \frac{.016t}{.016} \rightarrow t = \boxed{43.3 \text{ yrs.}}$

5. In a research experiment, a population of 100 fruit flies is increasing according to the law of exponential growth. After 2 days there are 300 flies.

   a. Find the value of k and write a function giving the population of fruit flies P in terms of the time t.

   $\frac{300}{100} = \frac{100e^{k(2)}}{100} \rightarrow 3 = e^{2k} \rightarrow \frac{\ln 3}{2} = \frac{2k}{2} \rightarrow \boxed{k = .549}$

   $\boxed{y = 100e^{.549t}}$

   b. How many flies will there be after 5 days?

   $y = 100e^{.549(5)} = \boxed{1556 \text{ flies}}$

   c. How long will it take for the number of fruit flies to equal 1000?

   $\frac{1000}{100} = \frac{100e^{.549t}}{100}$

   $10 = e^{.549t}$

   $\frac{\ln 10}{.549} = \frac{.549t}{.549}$

   $\boxed{t = 4.2 \text{ days}}$

6. Carbon-14 decays (changes to Carbon-12) according to an exponential decay model. The half-life of carbon-14 is 5715 years.

means 1/2 is left at the end.

a. Find the value of k. (Choose a starting amount)

$$\frac{50}{100} = \frac{100e^{k(5715)}}{100}$$

$$\frac{1}{2} = e^{5715k} \rightarrow \ln\frac{1}{2} = 5715k \rightarrow k = \frac{\ln\frac{1}{2}}{5715} = \boxed{-.000121}$$

b. Write an exponential function modeling carbon-14 decay.

$$y = Ce^{-.000121t}$$

c. Estimate the age of a bone which contains 42% of its original carbon-14.

$C = 100$
$y = 42$

$$\frac{42}{100} = \frac{\cancel{100}\, e^{-.000121t}}{\cancel{100}}$$

$$.42 = e^{-.000121t}$$

$$\frac{\ln .42}{-.000121} = \frac{-.000121t}{-.000121}$$

$$\boxed{t = 7169.4 \text{ yrs}}$$

Applications Involving Logarithmic Models

There are many examples of logarithmic models used in science and statistics. Some of these are:

- Sound intensity is measured in decibels (using a logarithmic scale).
- pH (measure of acidity or alkalinity) is measured using a logarithmic scale.
- The Richter Scale for measuring the intensity of earthquakes is a logarithmic scale.

Examples.

7. Students in a mathematics class were given an exam and then retested monthly with an equivalent exam. The average scores for the class are given by the human memory model: $S(t) = 78 - 17\log(t+1)$, $0 \le t \le 12$, where t is time in months.

a. What was the average score on the original exam?

$t = 0$

$$S = 78 - 17\log(1) = \boxed{78}$$

b. What was the average score after 3 months?

$t = 3$

$$S = 78 - 17\log(4) = \boxed{67.8}$$

c. What was the average score after 11 months?

$t = 11$

$$S = 78 - 17\log(12) = \boxed{59.7}$$

8. Chemists use the pH scale to test acidity. The equation is: $pH = -\log x$., where x = concentration of hydrogen ion in a solution.

a. The hydrogen ion concentration of a vinegar (acetic acid) solution is about $\underbrace{1.5\times10^{-5}}_{x}$. Find the pH of this solution.

$$pH = -\log(1.5\times10^{-5}) = \boxed{4.8}$$

b. Seawater has a pH of 8.5. Find the hydrogen ion concentration of seawater.

$$\frac{8.5}{-1} = \frac{+\log x}{-1}$$

$$-8.5 = \log x$$

$$10^{-8.5} = x$$

$$\boxed{x = 3.2\times10^{-9}}$$

9. On the Richter Scale, the magnitude R of an earthquake of intensity I is given by $\mathbf{R = \log I}$. (I is the wave energy of the earthquake, measured in joules.)

   a. Find the intensities of the San Francisco Bay area earthquake of 1989 which measured 7.1 on the Richter Scale and the San Francisco earthquake of 1906 which measured 8.6. Write each answer as a power of 10.

San Fran 1989

$7.1 = \log I$

$10^{7.1} = I$ $\boxed{I = 1.26 \times 10^7}$

San Fran 1906

$8.6 = \log I$

$10^{8.6} = I$ $\boxed{I = 3.98 \times 10^8}$

   b. Use a ratio to determine how many times more intense the 1906 earthquake was than the 1989 earthquake.

$\frac{3.98 \times 10^8}{1.26 \times 10^7} = \boxed{31.6 \text{ times Stronger}}$

Assignment 10.5

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Day 5

Unit 10 Review

Assignment 10.6

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Day 6

Unit 10 Test ***All late/absent assignments due for Unit 10***

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