Unit 11 Notes

Sequences & Series

Day 1

Arithmetic Sequences

An **arithmetic sequence** is a sequence of terms in which the difference between any two consecutive terms is a constant (**d = common difference**). Arithmetic sequences have explicit formulas to calculate each term of the sequence using the index (n), and a recursive formula for generating each new term based on a preceding term of the sequence.

Explicit Formula:	$a_n = a_1 + d(n-1)$
Recursive Formula:	$a_n = a_{n-1} + d$

Examples. Analyze each arithmetic sequence and identify the common difference. Then, find (or draw) the next three terms and write the explicit and recursive formulas.

a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_n

1. −5, −1, 3, 7, 11, 15, 19, 23 …

   d = 4

   explicit formula: $a_n = -5 + 4(n-1)$

   recursive formula: $a_n = a_{n-1} + 4$

a_1 a_2 a_3 a_4 a_5 a_6 a_7 … a_n

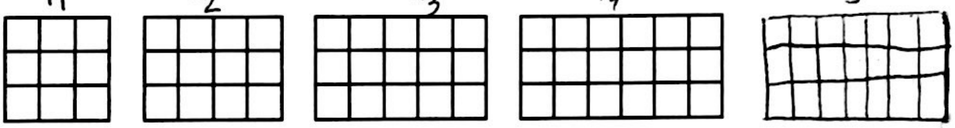
2. 27, 19, 11, 3, -5, -13, -21 …

   d = -8

   explicit formula: $a_n = 27 + -8(n-1)$

   recursive formula: $a_n = a_{n-1} - 8$

3. a_1 a_2 a_3 a_4 a_5



   d = 3

   explicit formula: $a_n = 9 + 3(n-1)$

   recursive formula: $a_n = a_{n-1} + 3$

You can use an explicit formula to find the value of the *n*th term of an arithmetic sequence when the previous term is unknown.

Examples.

4. Use the explicit formula to find the 25^{th} term in the sequence 5, 11, 17, 23, 29…

   $d = 6$ $a_1 = 5$ $a_n = 5 + 6(n-1)$

   $a_{25} = 5 + 6(25-1) = 5 + 6(24) = \boxed{149}$

5. Suppose you participate in a bike-a-thon for charity. The charity starts with $1100 in donations. Each participant must raise at least $35 in pledges. What is the minimum amount of money raised if there are 75 participants? (Find the explicit formula and use it to answer the question.)

   $d = 35$ $a_1 = 1100$ $a_n = 1100 + 35(n-1)$

   $a_{75} = 1100 + 35(75-1)$

   $a_{75} = 3,690$

   $3,690

Assignment 11.1

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Day 2

Arithmetic Series

A **series** is the sum of terms in a sequence. The sum of the first n terms of a sequence is denoted by S_n. For example, S_3 is the sum of the first three terms of a sequence. There is special notation for the summation of terms using a capital sigma, and looks like this:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n.$$

i is the indexing variable (can also be j, k, or n)
n is the position of the last term in the series
a_i is the explicit formula for the sequence and represents the terms.

A series can be *finite* or *infinite*. A **finite series** is the sum of a finite number of terms. An **infinite series** is the sum of an infinite number of terms.

Examples. Use sigma notation to rewrite each finite series, and then compute the sum.

$a_1 = 5 \quad d = 4 \quad a_i = 5 + 4(n-1)$

1. $5 + 9 + 13 + 17 + 21$ $\quad 5 + 4n$ $\quad 4n + 1$

$S_5 = \sum_{i=1}^{5} 4i + 1 = 5+9+13+17+21$
$= \boxed{65}$

$S_5 = 65$

alternate

2. ~~3 + 6 + 12 + 24 + 48 + 96 + 192~~

Not an arithmetic series

$S_7 =$

Finite arithmetic series can always be computed by adding each individual term, but this can take a lot of time. A famous mathematician named Carl Friedrich Gauss developed an easier way to compute these series. When Gauss was in elementary school, his teacher asked the class to calculate the sum of the first 100 positive integers. Gauss determined the answer in a matter of seconds. How did he do it?

Gauss's method can be generalized for any finite arithmetic series: $\boxed{S_n = \dfrac{n(a_1 + a_n)}{2}}$

Practice with previous example.

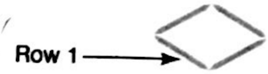
$S_5 = \dfrac{5(\overset{a_1}{5} + \overset{a_5}{21})}{2} = 65$

Neat!

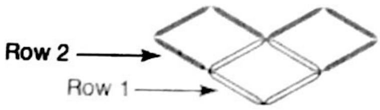
move ↓

Examples.

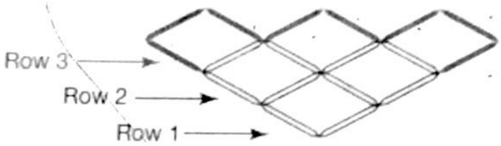
3. Josephine is helping her little brother Pauley with his latest art project. He is using toothpicks to create a tessellation (a tessellation is created when a geometric shape is repeated over a two-dimensional plane such that there are no overlaps and no gaps). Pauley starts his tessellation project by gluing toothpicks to a large piece of poster board to make a single diamond shape:



Then he places additional toothpicks parallel to the first row to create the second row. The second row consists of two diamond shapes:



He continues to place toothpicks in this manner, so that each row has one more diamond shape than the previous row:



a. Complete the table to show the number of additional toothpicks used to create each row:

Row	Number of Additional Toothpicks Used to Create the Row
1	4
2	6
3	8
4	10
5	12

b. What type of sequence is this? Write an explicit formula for this sequence. Let n represent the row number, and let a_n represent the number of additional toothpicks used to create that row.

Arithmetic. $a_n = 4 + 2(n-1) \Rightarrow a_n = 2n + 2$

c. Suppose Pauley knows that he wants his tessellation to include 10 rows. How many toothpicks will he need for just the 10th row?

$a_{10} = 2(10) + 2 \Rightarrow 20 + 2 = \boxed{22}$

d. Use sigma notation to represent the total number of toothpicks Pauley needs to complete 10 rows of his tessellation. Then, calculate this amount using Gauss' formula..

$a_{10} = 22$

$S_{10} = \sum_{i=1}^{10} 2i+2$ $\quad S_{10} = \frac{10(4+22)}{2} = 130$

e. What if Pauley decides to include 18 rows instead of 10? Will a box of 350 toothpicks be enough for his project?

$a_{18} = 2(18) + 2$
$36 + 2$
$= 38$

$S_{18} = \frac{18(4+38)}{2} = 378$

No, he'll be short, 28 toothpicks

4. Find the sum of the first 15 terms of the following arithmetic sequences. If an explicit formula is not given, find it.

a. $a_n = 12 + 3(n-1)$

$a_1 = 12 + \cancel{3(1-1)} \Rightarrow 12$

$a_{15} = 12 + 3(15-1) \Rightarrow 54$

$S_{15} = \frac{15(12+54)}{2} = 495$

b. 25, 19, 13, . . .

$a_1 = 25 \quad d = -6$

$a_n = 25 - 6(n-1)$

$a_{15} = 25 + -6(15-1)$

$a_{15} = -59$

$S_{15} = \frac{15(25 + -59)}{2} = -255$

$S_{15} =$

5. You are in charge of setting up for your high school band's annual Spring concert. The concert will be held outdoors on the school football field, and one of your duties is to arrange the seating for the show. You have room for 11 rows, and each row will have 3 more chairs than the previous row. The first row only has room for 7 chairs.

a. How many chairs can you put in the second row? third row? Write an explicit formula for this arithmetic sequence.

1st row: 7 2nd: 10 3rd: 13 $a_n = 7 + 3(n-1)$

$7 + 3n - 3$

b. What is the maximum number of people that can be seated? Use sigma notation to represent the total number of chairs and then calculate the maximum.

$S_{11} = \sum_{i=1}^{11} 3i + 4 \Rightarrow S_{11} = \frac{11(7+37)}{2} = 242$ seats

$a_{11} = 3(11) + 4 = 37$

c. What if you were able to fit 15 rows instead of 11? Represent the total number of chairs with sigma notation and calculate the number of people that can be seated in 15 rows.

$S_{15} = \sum_{i=1}^{15} 3i + 4 \Rightarrow S_{15} = \frac{15(7+49)}{2} = 420$ seats

Assignment 11.2 $a_{15} = 3(15) + 4 = 49$

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Day 3

Geometric Sequences

A **geometric sequence** is a sequence of terms in which the ratio between any two consecutive terms is a constant (**r = common ratio**). Geometric sequences also have explicit and recursive formulas.

Explicit Formula:	$a_n = a_1 \cdot r^{n-1}$
Recursive Formula:	$a_n = a_{n-1} \cdot r$

Examples. Analyze each geometric sequence and identify the common ratio. Then, find the next three terms (for 6 and 7) and write the explicit and recursive formulas.

1. 2, −10, 50, −250, 1250, -6250, 31250

$r =$ -5

explicit formula: $a_n = 2 \cdot (-5)^{n-1}$

recursive formula:

$a_n = a_{n-1} \cdot (-5)$

2. 9, 3, 1, $\frac{1}{3}$, 1/9, 1/27, 1/81...

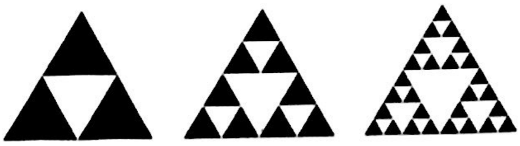
$r =$ 1/3

explicit formula: $a_n = 9 \cdot (1/3)^{n-1}$

recursive formula:

$a_n = a_{n-1} \cdot (1/3)$

3. number of black triangles



$r =$ 3

explicit formula: $a_n = 3 \cdot (3)^{n-1}$

recursive formula: $a_n = a_{n-1} \cdot (3)$

The explicit formula is helpful for finding a specific term without knowing the preceding term.

Examples.

4. For the geometric sequence 4, 10, 25, 62.5, … use the explicit formula to find the indicated term.

Explicit formula (*n*th term): $a_n = 4 \cdot (2.5)^{n-1}$

a. 5th term $a_5 = 4 \cdot (2.5)^{5-1} = 156.25$

b. 14th term $a_{14} = 4 \cdot (2.5)^{14-1} = 2,086,162.57$

5. Suppose you want a reduced copy of a photograph. The actual length of the photograph is 10 inches. The smallest size the copier can make is 64% of the original. Find the length of the photograph after five reductions at 64%.

$a_1 = 10 \quad r = .64$

$a_5 = 10(.64)^{5-1} \Rightarrow 1.68\text{ in}$

Assignment 11.3

Day 4

Finite Geometric Series

A **geometric series** is the sum of the terms of a geometric sequence. The formula to compute any finite geometric series is:

$$\boxed{S_n = \frac{a_1(1-r^n)}{1-r}}$$

Examples. Identify the number of terms n, the common ratio r, and the first term a_1 for each series. Then compute the sum using the formula.

1. $1+2^1+2^2+2^3+2^4$

$n=5 \quad r=2 \quad a_1=1$

$S_5 = \frac{1(1-2^5)}{1-2} = \boxed{31}$

2. $1+5+25+125+625$

$n=5 \quad r=5 \quad a_1=1$

$S_5 = \frac{1(1-5^5)}{1-5} = \boxed{781}$

3. $1-2+4-8+16-32$

$n=6 \quad r=-2 \quad a_1=1$

$S_6 = \frac{1(1-(-2)^6)}{1-(-2)} = -21$

4. $\sum_{i=1}^{8} 5^{i-1}$

$i=8 \quad r=5 \quad a_1=5^{1-1}=1$

$S_8 = \frac{1(1-5^8)}{1-5} = 97,656$

5. Jane analyzes the salary schedule for the same position at two different electrical engineering companies, Nothing's Shocking and High Voltage. The salary schedules for the first 5 years are provided with promises from each company that the **rate of salary increase** will be the same each year.

Time (years)	Nothing's Shocking Salary ($)	High Voltage Salary ($)
1	40,000	46,000
2	42,400	47,840
3	44,944	49,754
4	47,641	51,744
5	50,499	53,814

1.06   1.04

a. What is the salary in year 10 for each company? Write an explicit formula for each company's salaries.

N.S.: $a_n = 40,000(1.06)^{n-1}$ Yr. 10 = $67,580

H.V.: $a_n = 46,000(1.04)^{n-1}$ Yr. 10 = $65,472

b. Assuming all other factors are equal, which company offers the better salary over a 10-year period?

N.S. $S_{10} = \frac{40,000(1-1.06^{10})}{1-1.06} = \$527,232$

H.V. $S_{10} = \frac{46,000(1-1.04^{10})}{1-1.04} = \$552,281$

c. What would be the difference in total career salary if you work for 30 years?

N.S. $S_{30} = \frac{40,000(1-1.06^{30})}{1-1.06} = \$3,162,327$

-$582,420 difference

H.V. $S_{30} = \frac{46000(1-1.04^{30})}{1-1.04} = \$2,579,907$

Infinite Geometric Series

The formulas for finding the sum of arithmetic or geometric series only work if they are finite. However, sometimes an infinite geometric series can also have a finite sum. A geometric series converges if the common ratio of the series is between -1 and 1 and the sum exists. If not, then the series diverges and the sum is infinity. The formula to compute a geometric series when $-1 < r < 1$ is:

$$\boxed{S_\infty = \frac{a_1}{1-r}}$$

Examples. Determine whether each infinite geometric series will have a finite sum. If the series has a finite sum, use the formula to compute the sum.

6. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

finite $r = 1/3$

$S_\infty = \dfrac{1/3}{1-1/3} = 0.5$

7. $\sum_{i=1}^{\infty} \frac{1}{3}(3)^i$

$r = 3$ infinite

8. $\sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i$

finite $r = 1/4$

$S_\infty = \dfrac{1/4}{1-1/4} = 1/3$

Assignment 11.4

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Day 5

Applications of Arithmetic and Geometric Series

Examples.

1. Benjamin is anxious. After finishing his undergraduate degree, he must take the GRE in order to get into graduate school, but the amount of information he needs to cover is overwhelming. "I only have a month to prepare!" he exclaims. Sally tells him to calm down and start slowly. She recommends studying just 15 minutes the first day and adding 3 minutes each day to his study time. "But a friend told me that you need to study at least 20 hours to get ready for this thing! I think I need a different plan." Will Sally's plan lead to enough study time? Show all work and explain your reasoning.

15, 18, 21, ...

$d = 3$ $a_1 = 15$

$a_n = 15 + 3(n-1)$

$a_{30} = 15 + 3(30-1)$

$a_{30} = 102$ min

$S_{30} = \dfrac{30(15+102)}{2}$

$S_{30} = 1{,}755$

29.25 hrs.

2. Rhonda is considering two different physical therapist positions.
   - Range of Motion offers an initial salary of $50,000 per year with annual increases of $1,500 per year.
   - Mobility, Inc. offers an initial salary of $42,000 with a guaranteed 4% increase in salary every year.

   a. Is this situation arithmetic or geometric? Explain your reasoning.

RM: Arithmetic b/c your adding onto her salary

MI: Geometric b/c your multiplying her salary by 1.04

   b. Determine the years for which Range of Motion pays more than Mobility, Inc.

RM: $a_n = 50{,}000 + 1{,}500(n-1)$ $a_{10} = 63{,}500$ $a_{15} = 71{,}000$ $a_{14} = 69{,}500$ $a_{13} = 68000$

years 1-13

MI: $a_n = 42{,}000(1.04)^{n-1}$ $a_{10} = 59{,}779$ $a_{15} = 72{,}730$ $a_{14} = 69{,}933.$ $a_{13} = 67{,}243$

   c. Determine which company pays more salary over a 30-year career. Show all work and explain your reasoning.

RM: $a_1 = 50{,}000$ $a_{30} = 93{,}500$

$$S_{30} = \frac{30(50000 + 93{,}500)}{2} = \$2{,}152{,}500$$

MI:

$$S_{30} = \frac{42000(1 - 1.04^{30})}{1 - 1.04} = \$2{,}355{,}567$$

3. A stomach virus spreads rapidly through a town. Initially only 12 people were infected, but the virus spreads quickly, increasing the number of people infected by 15% each day.

   a. How many new people are infected on the 10th day? Show all work and explain your reasoning.

$a_n = 12 \cdot (1.15)^{n-1}$ $a_{10} = 42$ geometric sequence $r = 1.15$

42 ppl on day 10.

   b. How many total people are infected on the 10th day? Show all work and explain your reasoning.

$$S_{10} = \frac{12(1 - 1.15^{10})}{1 - 1.15} = 244 \text{ ppl.}$$

Assignment 11.5

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Day 6

Unit 11 Review

Assignment 11.6

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Day 7

Unit 11 Test ***All late/absent assignments due for Unit 11***

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