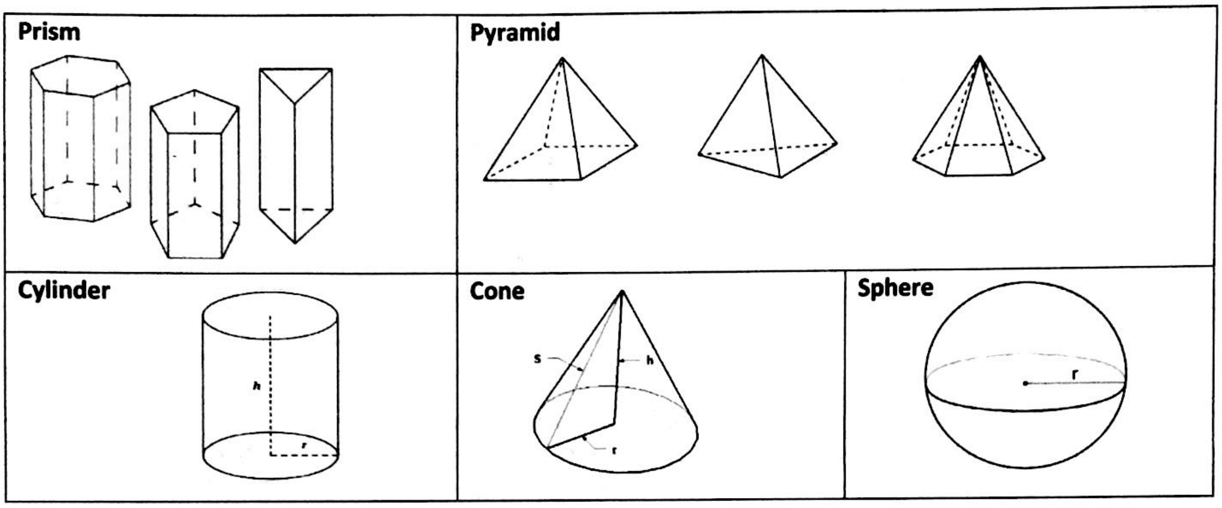
Unit 3
Polynomial Functions of Higher Degree

Day 1
Volume of Three-dimensional Geometric Figures



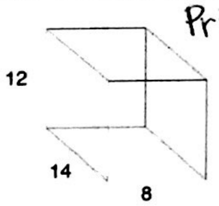
Volume: The amount of space occupied by a 3-dimensional object.

Volume Formulas:

Prism = $(area\ of\ base)(height)$ Pyramid = $\frac{1}{3}(area\ of\ base)(height)$ Sphere = $\frac{4}{3}\pi r^3$

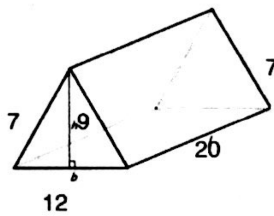
Cylinder = $(\pi r^2)(height)$ Cone = $\frac{1}{3}(\pi r^2)(height)$

Examples. Find the volume of the following:

1. 

Prism

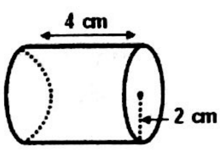
$(8\cdot14)(12) = \boxed{1344}$

2. 

Prism

$\left(\frac{1}{2}(12\cdot9)\right)(20)$

$= \boxed{1080}$

3. 

Cylinder

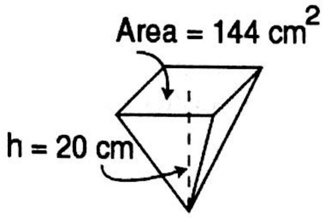
$\pi(2^2)(4) = 16\pi$

$\approx \boxed{50.3\ cm^3}$

4. A company produces 100,000 square based pyramid shaped popcorn containers every day, such as the one in the diagram below. (The top of the popcorn container is open so you can fill it with popcorn.) What volume of popcorn can each container hold?

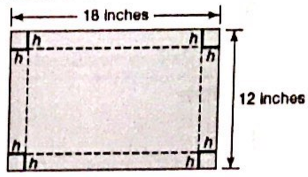
Pyramid

$\frac{1}{3}(144)(20) = \boxed{960\ cm^3}$



Business is Growing

The Plant-A-Seed Planter Company produces planter boxes. To make the boxes, a square is cut from each corner of a rectangular copper sheet. The sides are bent to form a rectangular prism without a top. Cutting different sized squares from the corners results in different sized planter boxes. Plant-A-Seed takes sales orders from customers who request a sized planter box.



Each rectangular copper sheet is 12 inches by 18 inches. In the diagram, the solid lines indicate where the square corners are cut and the dotted lines represent where the sides are bent for each planter box.

5. Organize the information about planter boxes of various sizes.
   a. Complete the table.
   b. What patterns do you notice in the table?

Corner Length	Height (in)	Width (in)	Length (in)	Volume (in)
0	0	18	12	0
1	1	16	10	160
2	2	14	8	224
3	3	12	6	216
h	h	18−2h	12−2h	h(18−2h)(12−2h)

6. Analyze the relationship between the height, length, and width of each planter box.
   a. What is the largest sized square corner that can be cut to make a planter box? Explain your reasoning.

      Can't be more than 6 in. or there won't be a base.

   b. What is the relationship between the size of the corner square and the length and width of each planter box?

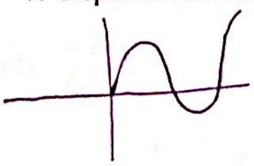
      h

      $W = 18 - 2h$ $\quad$ $l = 12 - 2h$

   c. Write a function *V(h)* to represent the volume of the planter box in terms of the corner side of length *h*.

      $V(h) = l \cdot w \cdot h$
      $= (12-2h)(18-2h)(h)$

7. Represent the function on a graphing calculator using the window [−10,15] by [−400,400].



   a. What is the maximum volume of a planter box? State the dimensions of this planter box.

      228 in.^{3} $\quad$ (x, y)
      height volume
      height = 2.35 plug it in for h in 6b.
      7.3 in × 13.3 in × 2.35 in.

   b. Identify the domain of the function *V(h)*. Is the domain the same or different in terms of the context of this problem?

      ℝ
      x is between 0 & 6 in.

   c. Identify the range of the function *V(h)*. Is the range the same or different in terms of the context of this problem?

      ℝ
      y is between 0 & 228 in.

   d. What do the x-intercepts represent in this problem situation? Do these values make sense in terms of this problem situation?

      0, 6, 9
      Where the volume = 0.
      No, because there won't be a planter if V = 0.

Using Polynomial Functions to Represent Volumes

Example.

8. Three forms of a volume function are shown:

A. $V(x) = \underbrace{(x+2)(3x-2)}_{\text{FOIL}}(x+4)$

$= (3x^2 - 2x + 6x - 4)(x+4)$

$= (3x^2 + 4x - 4)(x+4)$

$= 3x^3 + 12x^2 + 4x^2 + 16x - 4x - 16$

$= 3x^3 + 16x^2 + 12x - 16$ ✓

B. $V(x) = (x+2)(3x^2 + 10x - 8)$

$= 3x^3 + 10x^2 - 8x + 6x^2 + 20x - 16$

$= 3x^3 + 16x^2 + 12x - 16$ ✓

C. $V(x) = 3x^3 + 16x^2 + 12x - 16$ ✓

Are the three above functions different or equivalent? Determine algebraically whether these functions are equivalent by multiplying and combining like terms. Verify graphically with your graphing calculator. ✓

A, B & C are all equivalent

Assignment 3.1

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Day 2

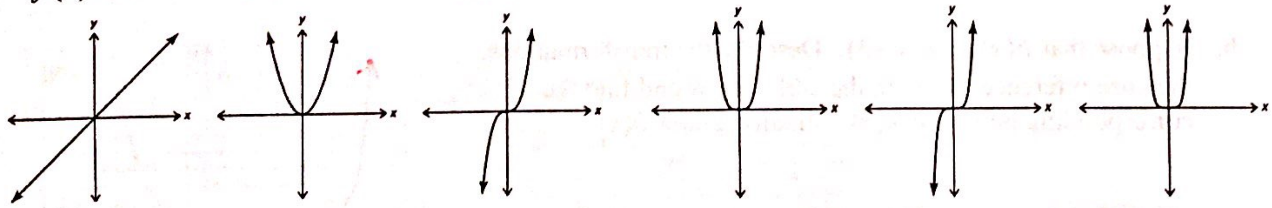
Power Functions

A common type of polynomial function is a *power function*. A **power function** is a function of the form $P(x) = ax^n$, where n is a natural number.

The **end behavior** of a graph of a function is the behavior of the graph as x approaches infinity and as x approaches negative infinity.

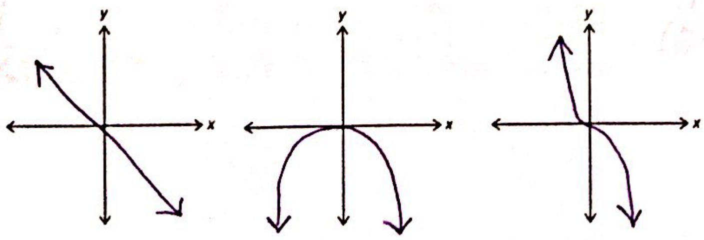
Consider each power function and its graph in the sequence shown.

$f(x) = x$ $\quad f(x) = x^2$ $\quad f(x) = x^3$ $\quad f(x) = x^4$ $\quad f(x) = x^5$ $\quad f(x) = x^6$



Sketch the graph of the following functions:

$g(x) = -x$ $\quad g(x) = -x^2$ $\quad g(x) = -x^3$



Complete the table to describe the end behaviors for any power function.

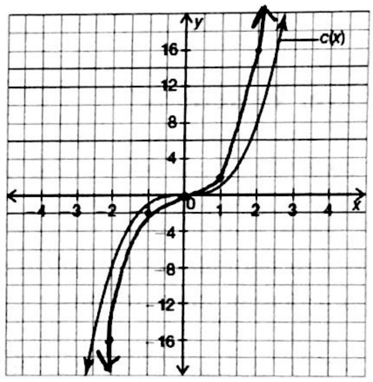
Degree	Odd Ends opposite	Even Ends same
Leading coefficient	Positive Right side ↑	Negative Right side ↓

Transformations of Power Functions

In Unit 2 you learned about rigid transformations (horizontal and vertical shifts/reflections) and dilations (vertical/horizontal stretches/compressions). These transformations can be applied to polynomial functions as well as quadratic functions.

Examples.

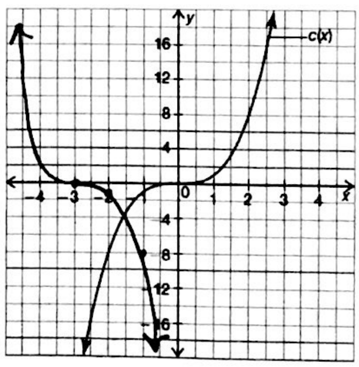
1. The graph of the basic cubic function $c(x) = x^3$ is shown.
   a. Suppose that $g(x) = 2c(x)$. Describe the transformation, then use reference points in the table below and find the corresponding points on $g(x)$. Finally, graph $g(x)$.



Reference Points on c(x)	→	Corresponding Points on g(x)
(0, 0)	→	(0,0)
(1, 1)	→	(1, 2)
(2, 8)	→	(2, 16)

Vertical stretch of 2

   b. Suppose that $h(x) = -c(x+3)$. Describe the transformations, then use reference points in the table below and find the corresponding points on $h(x)$. Finally, graph $h(x)$.



Reference Points on c(x)	→	Corresponding Points on h(x)
(0, 0)	→	(-3, 0)
(1, 1)	→	(-2, -1)
(2, 8)	→	(-1, -8)

Reflect across x axis
Left 3

2. Describe the effects the following transformations have on the basic function $c(x)=x^3$. Then determine the point on the new function that corresponds to the point (2, 8) on $c(x)$

$a(x)=c(x-5)$

right 5

(7,8)

$j(x)=c(-x)+2$

reflect across y
up 2

(-2,10)

$m(x)=c(x+2)-3$

left 2
down 3

(0,5)

$b(x)=-c(x)+5$

reflect across x
up 5

(2,-3)

$k(x)=c(\frac{1}{2}x)$

Horizontal stretch by 2

(4,8)

$n(x)=3c(x-2)+4$

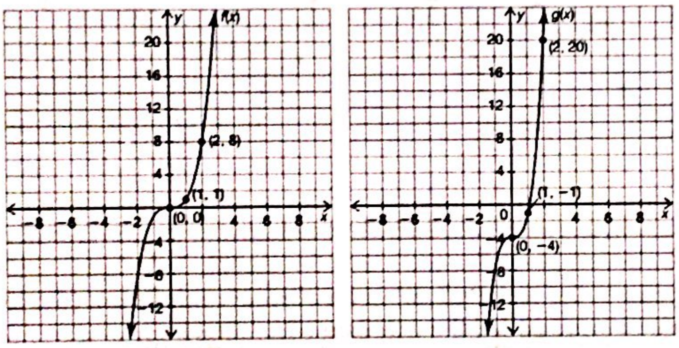
Vertical stretch by 3
right 2
up 4

(4,28)

Writing equations for transformed functions:

3. Analyze the graphs of $f(x)$ and $g(x)$. Describe the transformations performed on $f(x)$ to create $g(x)$. Then write an equation for $g(x)$ in terms of $f(x)$.

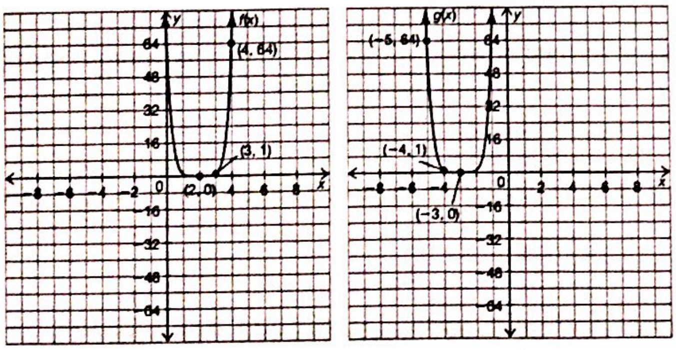
a.



down 4
V. Stretch 3

$g(x) = 3f(x) - 4$

b.



Left 5

$g(x)= f(x+5)$

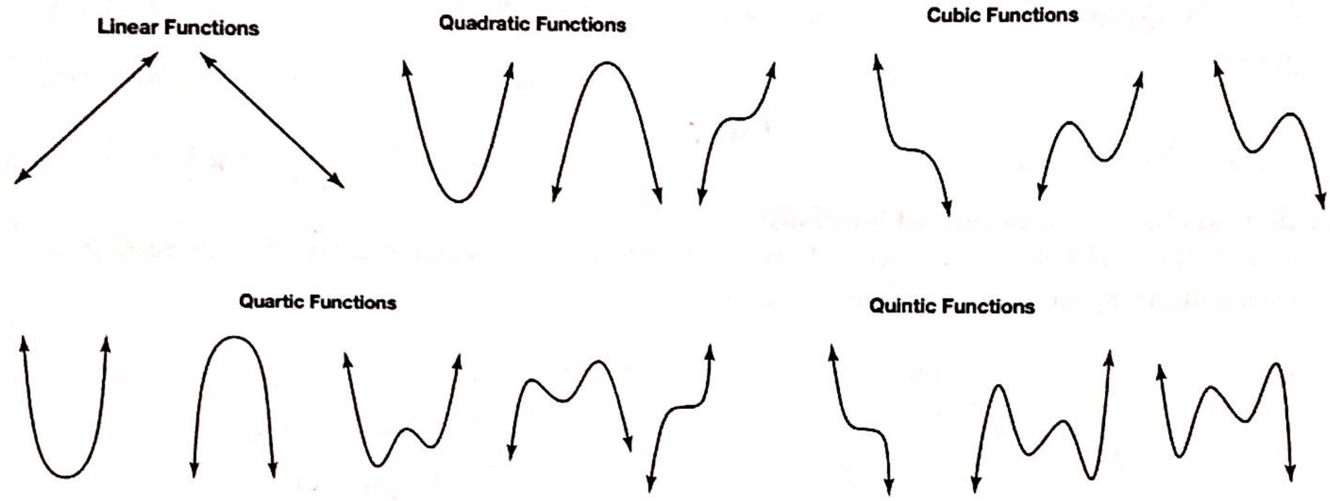
Day 3

Key Characteristics of Polynomial Functions

Polynomial graphs are:

- **Smooth**
- **Continuous**
- **End behaviors are determined by the behavior of the degree and the leading coefficient (*a*)**

Possible shapes of basic polynomial functions are shown below. These graphs often have turning points, where a polynomial function changes from rising to falling or from falling to rising.



Where a polynomial function has a turning point there is a **relative maximum** (highest point in a section of the graph) or a **relative minimum** (the lowest point in a section of the graph). Some polynomial graphs have an **absolute maximum** (the highest point in the entire graph) or an **absolute minimum** (the lowest point in the entire graph). The set of all these points are called the **extrema** of a function.

What conclusions can you make about the end behavior of all even degree polynomial functions?

Same direction

What conclusions can you make about the end behavior of all odd degree polynomial functions?

opposite direction

What conclusions can you make about the domain and range of all even degree polynomial functions?

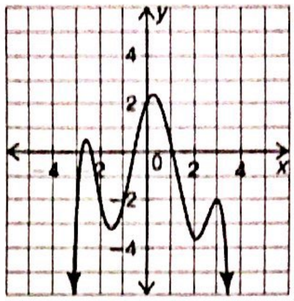
D: $\mathbb{R}$ R: Limited

What conclusions can you make about the domain and range of all odd degree polynomial functions?

D: $\mathbb{R}$ R: $\mathbb{R}$

Examples.

1. Consider the graph shown.



   a. Is the a-value of this function positive or negative?

      negative

   b. Is the degree of this function even or odd?

      even

   c. State the domain of this function.

      $\mathbb{R}$

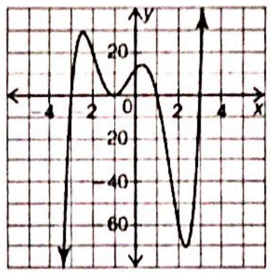
   d. State the range of this function.

      $y \leq 2.3$ (about)

   e. Determine the number of relative extrema and the number of absolute extrema in the graph.

      4   1

2. Consider the graph shown.



   a. Is the a-value of this function positive or negative?

      Positive

   b. Is the degree of this function even or odd?

      odd

   c. State the domain of this function.

      $\mathbb{R}$

   d. State the range of this function.

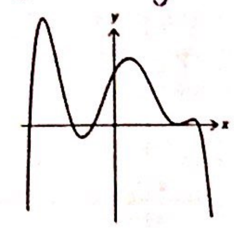
      $\mathbb{R}$

   e. Determine the number of relative extrema and absolute extrema in this graph.

      4   None

3. Analyze each graph. Circle the function(s) which **could** model the graph. Describe your reasoning to either eliminate or choose each function.

a. even, negative



$f_1(x) = -3x^5 - 2x^2 + 4x + 7$

not even

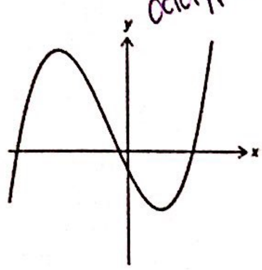
$f_2(x) = (x+2)(x+1.5)(x-2.5)^2$

not negative

$f_3(x) = -3x^4 - 2x^2 + 4x + 7$

even & ~~positive~~ negative

b. odd, positive



$f_1(x) = 2(x+7)(x+1)(x-5)$

odd & positive

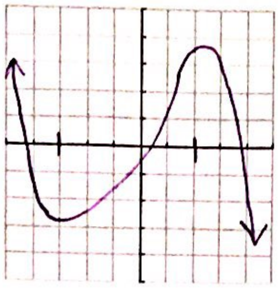
$f_2(x) = -3(x+7)(x+1)(x-5)$

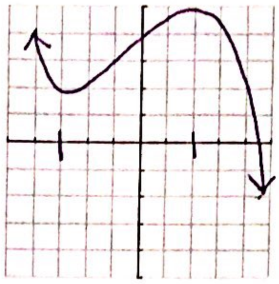
negative

$f_3(x) = (x+7)(x+1)(x-5)(x-3)$

even

4. Sketch graphs for two different polynomial function with the given characteristics: as $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to \infty$; there is a relative maximum at $x = 2$ and a relative minimum at $x = -3$

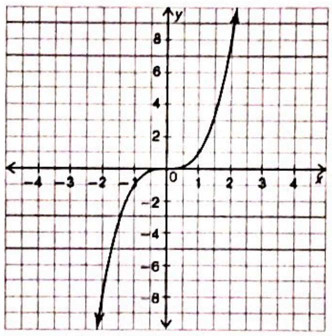




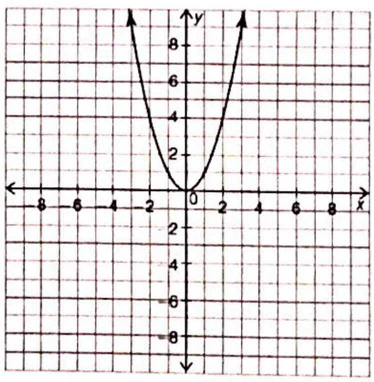
Symmetry

An axis of symmetry divides a graph into two parts that are mirror images of each other. If you do a reflection across a line and the graph looks exactly the same as the original, it means that the graph is symmetric with respect to that line.

5. Analyze the graph shown.
   a. Identify 2 symmetric points.
      (2,8) (-2,-8)
   b. If one point is (a, b), what are the coordinates of the other symmetric point?
      (a,b) (-a,-b)



6. Analyze the graph shown.
   a. Is there a line of symmetry? Do you see any symmetry?
      yes
   b. Identify two symmetric points.
      (2,4) (-2,4)
   c. If one point is (a, b), what are the coordinates of the other symmetric point?
      (a,b) (-a,b)



An **even** function has a graph symmetric about the y-axis, so $f(-x) = f(x)$.

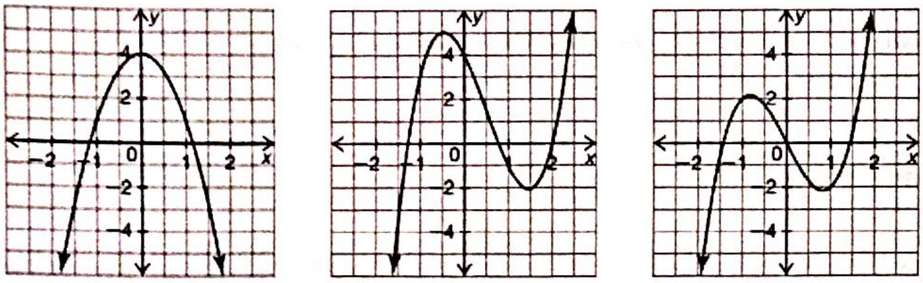
Plug in (-x) for x

An **odd** function has a graph symmetric about the origin, so $f(-x) = -f(x)$.

Plug in (-x) for x & (-y) for y

7. Examine the graphs. Do the functions have even or odd symmetry?

a. **b.** **c.**



a. even b. neither c. odd

Sometimes it's hard to tell whether a function has even or odd symmetry, especially without a graph. To find this symmetry algebraically, find and simplify $f(-x)$. If $f(-x) = f(x)$ then the function has **even** symmetry. If $f(-x) = -f(x)$ the function has **odd** symmetry. If neither statement is true, then the function has **neither**.

8. Determine algebraically whether the functions are even, odd, or neither.

a. $f(x) = 2x^3 - 3x$ not even

$= 2(-x)^3 - 3(-x)$

$= -2x^3 + 3x$

$-y = 2(-x)^3 - 3(-x)$

$-y = -2x^3 + 3x$

$y = 2x^3 - 3x$ odd

b. $g(x) = 6x^2 + 10$

$= 6(-x)^2 + 10$ same

$= 6x^2 + 10$

even

c. $h(x) = x^3 - 2x + 7$ not even

$= (-x)^3 - 2(-x) + 7$

$= -x^3 + 2x + 7$

not odd

$-y = -x^3 + 2x + 7$

$y = x^3 - 2x - 7$

neither

Assignment 3.3

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Day 4

Building Cubic Functions and Sketching Graphs

The degree of a function determines the end behaviors (along with the leading coefficient), but the general shape of the function is determined by the zeros and factors of the functions. If the zeros are imaginary, they will not affect the shape of a graph, but if the zeros are real, each zero gives an x-intercept for the graph. From the Fundamental Theorem of Algebra, remember that there are always n zeros for a function of degree n. These zeros could be repeated. Imaginary/complex zeros must come in pairs.

Examples. Write a cubic function with the following zeros. There should be no i's in the final answer.

1. 2, 0, −4

$f(x) = x(x-2)(x+4)$

2. 0, $2i$, $-2i$

$f(x) = x(x-2i)(x+2i)$

$= x(x^2 + \cancel{2xi} - \cancel{2xi} - 4i^2)$

$= x(x^2 - 4i^2)$

$f(x) = x(x^2 + 4)$

3. 6 (multiplicity of 2), −5

$f(x) = (x-6)^2(x+5)$

Sketching Graphs of Polynomial Functions using *x*-intercepts

Zeros, Factors, and x-intercepts:
If $x = a$ is a real zero or root of a polynomial function, the following relationships exist:

- $f(a) = 0$
- $(x - a)$ is a factor of $f(x)$
- $(a, 0)$ is an x-intercept of the graph of $f(x)$

Finding Real Zeros: ① set function = 0 ② factor ③ set each factor = 0 & solve.

Multiplicity of Zeros: Power of factor

$(x-2)$ m: 1 $\quad (x-2)^2$: m: 2 $\quad (x-3)^3$ m: 3

Examples. Find all real zeros of the following functions by factoring and using the zero property. Then determine the multiplicity of each zero.

4. $f(x) = x^2 - 16$

$0 = (x-4)(x+4)$

$x - 4 = 0 \quad x + 4 = 0$

$\boxed{x = 4,\ m{:}1} \quad \boxed{x = -4,\ m{:}1}$

5. $f(x) = x^4 - 12x^3 + 36x^2$

$0 = x^2(x^2 - 12x + 36)$

$= x^2(x-6)(x-6)$

$= x^2(x-6)^2$

$\boxed{x = 0 \ m{:}2,\quad x = 6 \ m{:}2}$

6. $f(x) = x^4 - 5x^2 + 4$

$0 = (x^2 - 1)(x^2 - 4)$

$0 = (x+1)(x-1)(x+2)(x-2)$

Zeros: -1, 1, -2, 2
m:1 m:1 m:1 m:1

7. $f(x) = (x-3)(x^2 - 9)$

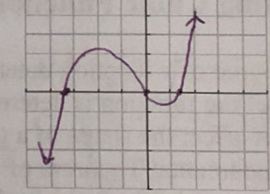
$0 = (x-3)(x-3)(x+3)$

Zeros: 3, -3
m:2 m:1

Using a calculator to find zeros:

8. Find the approximate value of the real zeros of $f(x) = x^3 + 2x^2 - 4x$ (round to 1 decimal place). Sketch a graph of the function.

Zeros: -3.2, 0, 1.2



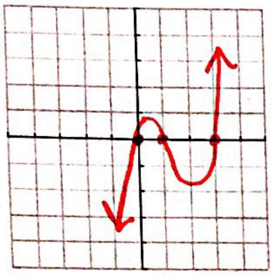
Graphing Polynomial Functions using *x*-intercepts and end behaviors

- **Characteristics of Polynomial Graphs:** Smooth - no sharp turns
  Continuous - no gaps
- **End behaviors depend on:**
  Degree and sign of the leading coefficient
- ***x*-intercepts:**
  The graph of a polynomial function will cross the *x*-axis at each *x*-intercept with an odd multiplicity. The graph will touch the *x*-axis at each *x*-intercept with an even multiplicity (extremum).
- ***y*-intercept:**
  Polynomial graphs have one y-intercept. Find it by putting in 0 for *x*.

If the multiplicity is odd it crosses.
If the multiplicity is even it bounces.

Examples. Factor to determine the zeros (x-intercepts) and multiplicities, use the degree to determine the end behaviors and evaluate the y-intercept. Then sketch the graph of the function.

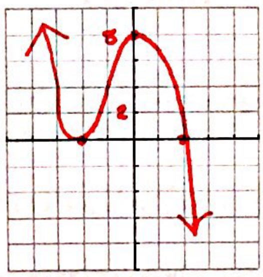
9. $h(x) = x^3 - 4x^2 + 3x = x(x^2-4x+3)$
$x(x-3)(x-1)$



Zeros: 0 m: 1 cross
3 m: 1 cross
1 m: 1 cross

Leading Coeff: 1 (positive)
Degree: 3 (odd)
End Behavior: ↓↑

10. $f(x) = -(x+2)(x^2-4)$
$-(x+2)(x-2)(x+2)$

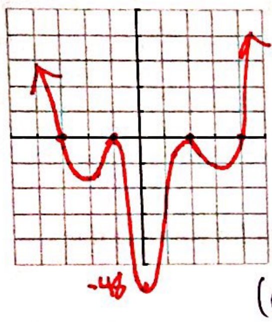


Zeros:
-2 m: 2 B
2 m: 1 C

LC: neg
Deg: odd
EB: ↑↓

Y-int: $-(0+2)(0^2-4)$
$(0,8)$ $-(2)(-4) = 8$

11. $f(x) = (x^2-x-2)^2(x^2-x-12)$
$= (x-2)^2(x+1)^2(x-4)(x+3)$

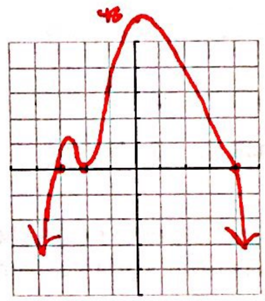


Zeros: 2 m: 2 B
-1 m: 2 B
4 m: 1 C
-3 m: 1 C

LC: +
Deg: even
EB: ↑↑

y-int: $(0^2-0-2)^2(0^2-0-12)$
$(0,-48) = (4)(-12)$
$= -48$

12. $g(x) = -(x+2)^2(x-4)(x+3)$



Zevos: -2 m: 2 B
4 m: 1 C
-3 m: 1 C

LC: −
Deg: even
EB: ↓↓

y-int: $-(0+2)^2(0-4)(0+3)$
$(0,48) = -(4)(-4)(3)$
$= 48$

13. Sketch a graph of a polynomial function that is a fifth-degree (odd) with two real zeros and a negative leading coefficient.

(There are many correct answers.)

EB: ↑↓

2 x-int.

Assignment 3.4

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Day 5
Unit 3 Review
Assignment 3.5

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Day 6
Unit 3 Test ***All late/absent assignments due for Unit 3***

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