# Unit 7
# Trigonometric Functions

*Day 1*
## Solving Trig Equations

"Backward trig": finding the angles when given a trig value. Use special triangles to find the reference angle and then ASTC to determine the quadrants and thus the angles, or the unit circle for quadrant angles.

*Examples.* Degrees
Find all angles $\theta$, $0^\circ \le \theta < 360^\circ$ that satisfy the given conditions.

1. $\tan\theta = -\sqrt{3}$

draw triangle
or use hand trick.


RA: $60^\circ$
QII or IV

$\boxed{120^\circ}$ $\boxed{300^\circ}$

2. $\cos\theta = \frac{\sqrt{2}}{2}$

RA: $45^\circ$ QI QIV

$\boxed{45^\circ \text{ \& } 315^\circ}$

Find all angles $x$, $0 \le x < 2\pi$, that satisfy the given conditions.
Radians

★Reminder Hand trick rationalizes the $45^\circ$ & $\pi/4$ trig ratios for sin & cos!

3. $\sin x = -\frac{1}{2}$

RA: $\pi/6$ Sin neg in QIII & QIV

$\boxed{5\pi/6 \text{ \& } 11\pi/6}$

4. $\sec x = \pm\sqrt{2}$

$\cos x = \pm\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

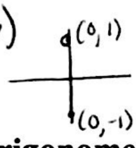
RA: $\pi/4$ All Quadrents

$\boxed{\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4}$

Find all values of $\theta$ or $x$ in the given interval.

5. $\cos x = 0$ $\quad 0 \le x < 2\pi$

(Quadrent angle)




$\boxed{\pi/2 \text{ \& } 3\pi/2}$

6. $\sin\theta = .396$ $\quad 0^\circ \le \theta < 360^\circ$

Use calc. (Degrees)

$\theta = \sin^{-1}(.396) = 23^\circ$

## Solving trigonometric equations

Use algebra techniques (including factoring, when necessary) to isolate the trig expression and "backward trig" to determine the angles.

Radians

*Examples.* Solve the following trigonometric equations; find all solutions in the interval $[0, 2\pi)$.

7. $2\cos x - 1 = 0$

$+1 \quad +1$

$\frac{\cancel{2}\cos x}{\cancel{2}} = \frac{1}{2}$

$\cos x = \frac{1}{2}$

QI IV

RA: $\pi/3$

$\boxed{\pi/3 \text{ \& } 5\pi/3}$

8. $\sin x - \sqrt{2} = -\sin x$

$+\sin x \quad +\sin x$

$2\sin x - \sqrt{2} = 0$

$+\sqrt{2} \quad +\sqrt{2}$

$\frac{2\sin x}{2} = \frac{\sqrt{2}}{2}$

$\sin x = \frac{\sqrt{2}}{2}$

QI & II

RA: $\pi/4$

$\boxed{\pi/4 \text{ \& } 3\pi/4}$

9. $6\tan^2 x - 2 = 0$

$+2 \quad +2$

$\frac{\cancel{6}\tan^2 x}{\cancel{6}} = \frac{2}{6}$

$\sqrt{\tan^2 x} = \sqrt{\frac{1}{3}}$

$\tan x = \pm\frac{1}{\sqrt{3}}$

All quadrents

RA: $\pi/6$

$\boxed{\pi/6,\ 5\pi/6,\ 7\pi/6,\ 11\pi/6}$

10. $\cot x\cos^2 x - \cot x = 0$

$\cot x(\cos^2 x - 1) = 0$

two eq.

$\cot x = 0 \Rightarrow \tan x = \frac{1}{0}$ or $= -\frac{1}{0}$ $\quad \frac{y}{x}$

$\boxed{x = 3\pi/2,\ \pi/2}$

$\cos^2 x - 1 = 0$

$+1 \quad +1$

$\sqrt{\cos^2 x} = \sqrt{1}$

$\cos = \pm 1$

$\boxed{x = 0, \pi}$

11. $2\sin^2 x - \sin x - 1 = 0$

$(2\sin x + 1)(\sin x - 1) = 0$

$\sin x = -\frac{1}{2}$ $\quad \sin x = 1$

Q III IV

RA: $\pi/6$

$\boxed{x = 7\pi/6, 11\pi/6}$ $\quad \boxed{x = \pi/2}$

12. $\sqrt{3}\sec x = -2$

$\frac{\sqrt{3}\sec x}{\sqrt{3}} = \frac{-2}{\sqrt{3}}$

$\sec x = -\frac{2}{\sqrt{3}}$

$\cos x = -\frac{\sqrt{3}}{2}$ $\quad \boxed{x = 5\pi/6, 7\pi/6}$

Q II III RA: $\pi/6$

For trigonometric equations which cannot be solved using algebraic methods, graphing in a calculator can be used to get approximate answers. Make sure to use a window which will show only the solutions that you want.

*Example:*

13. Use a calculator to find the solutions to $2\cos^2 x = \sin x$ in the interval [−180°, 180°]. Round answers to the nearest degree.

xmin xmax

$2\cos^2 x = \sin x$

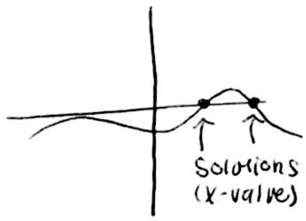
$-2\cos^2 x \quad -2\cos^2 x$

$0 = \sin x - 2\cos^2 x$

Graph: $y = \sin x - 2\cos^2 x$

or $y = 2\cos^2 x$

$y = \sin x$




$x = 51°, 128°$

Calculator:
2nd → calc
zero → helps you find x-ints.

Assignment 7.1

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*Day 2*

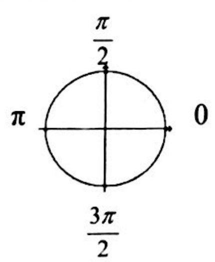
## Graphing Sine and Cosine Functions

Sine and cosine graphs have a shape called sinusoidal curves. Many applications of science (sound waves, for example) are modeled using sines or cosines.

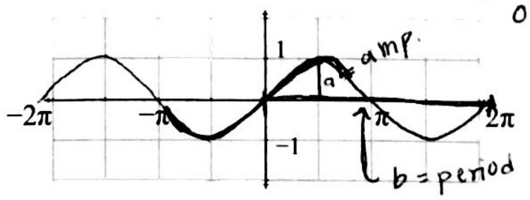
To graph a sine or cosine function, $x$ values represent angles (measured in radians), and $y$ values represent trig ratios. Using only the sines and cosines for quadrant angles (as seen on a unit circle), you can form a fairly accurate sketch for one period (cycle) of either a sine or cosine graph. The graphs should be continuous and smooth (no sharp turns).

Use the unit circle shown to sketch graphs of $y = \sin x$ and $y = \cos x$.

sin makes an "S" @ the origin

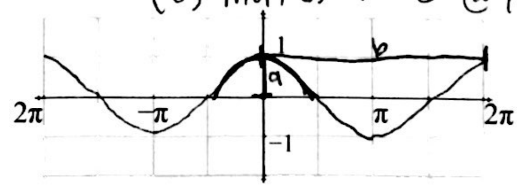



$f(x) = \sin x$




cos makes a "C" @ the origin

$f(x) = \cos x$



All trig functions have graphs that are periodic: they repeat the same pattern (called a period or cycle) to the right and to the left, forever. The sine and cosine both have the same shape (sinusoidal). Sine graphs are symmetric about the origin, while cosine graphs are symmetric about the $y$-axis, unless there is a shift.

All sinusoidal graphs have an amplitude and a period.

**Amplitude:** vertical distance from midline to the top of the graph.

**Period:** horizontal distance needed to complete one period (cycle) of the graph.

Before graphing, always determine the amplitude and period of the function.

period — horizontal shift

**Graphing** $y = a\sin(b(x-c)) + d$ or $y = a\cos(b(x-c)) + d$

amplitude — vertical shift

***Determine the amplitude and period before graphing:*** **amplitude** = $|a|$ **period** = $\dfrac{2\pi}{b}$

***Always label the scale on the x and y axes.***

negative amps cause a reflection.

**Graphing amplitude changes and/or reflections**

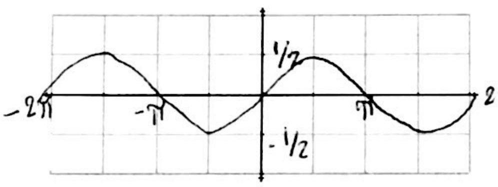
$a$ gives *amplitude* (vertical stretch/shrink) as well as *reflections* (when $a < 0$).

$\boxed{|a| = \text{amplitude}}$

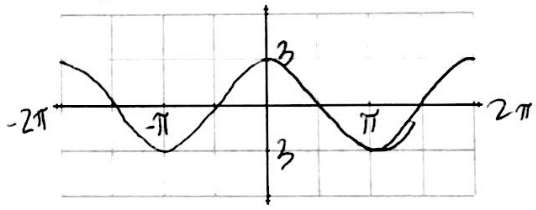
one positive one negative

*Examples.* Graph two periods of the following functions:

1. $f(x) = \frac{1}{2}\sin x$ $a = 1/2$ $p = 2\pi$




2. $y = 3\cos x$ $a = 3$ $b = 2\pi$

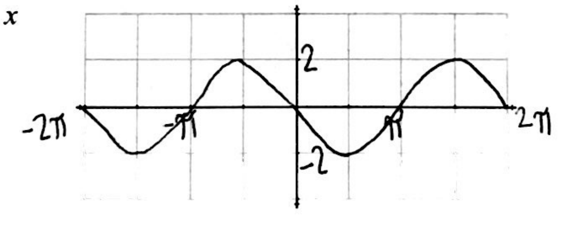



3. $f(x) = -2\sin x$

$a = 2$

$b = 2\pi$

Reflect over x-axis.




**Graphing period changes**

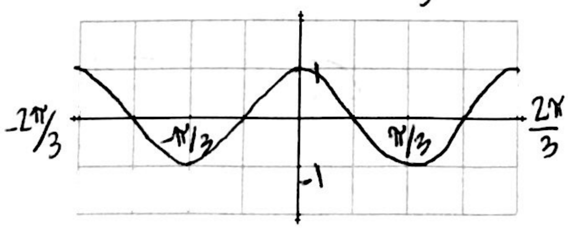
$b$ determines the *period* (horizontal stretch/shrink)

$\boxed{\dfrac{2\pi}{|b|} = \text{period}}$ $2\pi \div b$

b can be a fraction!

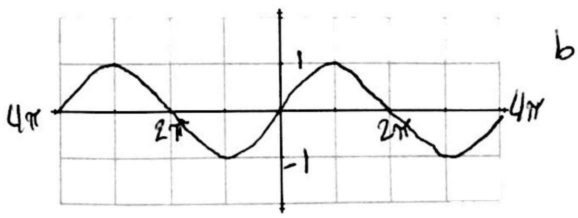
*Examples.* Graph two periods of the following functions:

4. $y = \cos(3x)$ $a = 1$ $b = \frac{2\pi}{3}$




5. $y = \sin\left(\frac{1}{2}x\right)$ $a = 1$ $b = \frac{2\pi}{1/2} = \frac{2\pi}{1} \div \frac{1}{2} \Rightarrow \frac{2\pi}{1} \cdot 2$

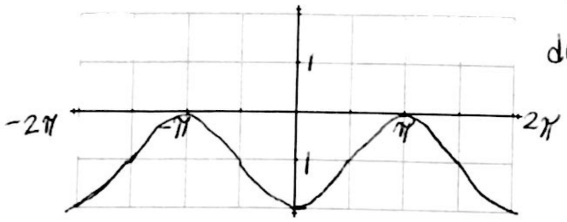
$b = 4\pi$



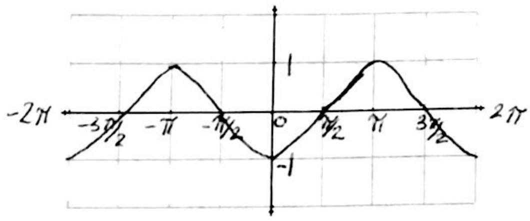
**Graphing horizontal/vertical shifts**
*d* results in a **vertical shift;** *c* results in a **horizontal shift** *(called a phase shift).*

*Examples.* Graph two periods of the following functions:

6. $y = -\cos x - 1$ $a=1$ $b=2\pi$ reflect over x-axis down 1

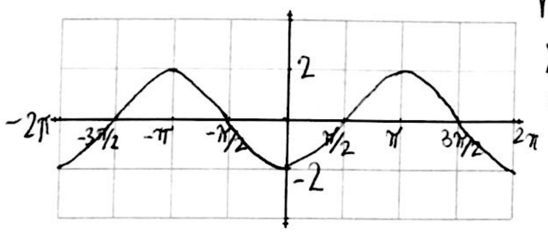


7. $y = \sin\left(x - \frac{\pi}{2}\right)$ $a=1$ $b=2\pi$ right $\pi/2$

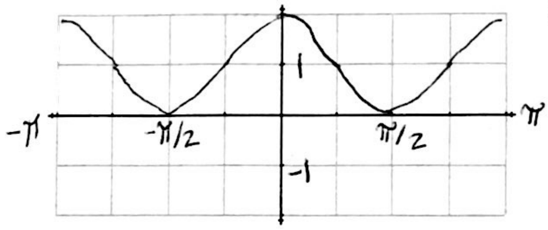


*Examples.* Now try graphing more than one transformation.

8. $f(x) = -2\sin\left(x + \frac{\pi}{2}\right)$ $a=2$ $b=2\pi$ reflect over x-axis left $\pi/2$



9. $g(x) = \cos(2x) + 1$ $a=1$ $b=\frac{2\pi}{2}=\pi$ up 1



Assignment 7.2

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*Day 3*

## Law of Sines

So far, you have solved sides and angles on right triangles. You will now learn two methods for solving oblique triangles (triangles with no right angle). To solve oblique triangles, you need to know the measure of at least one side and the measures of two other parts of the triangle. There are four possible cases for what you are given:

| You will use **Law of Sines** when you know: | You will use **Law of Cosines** when you know: |
|---|---|
| 1. Two angles and any side (AAS or ASA). | 3. Two sides and their included angle (SAS). |
| 2. Two sides and an angle opposite one of them (SSA). | 4. Three sides (SSS) |

Case 2 is tricky. It is known as the ambiguous case for the Law of Sines.

Important facts to keep in mind as you set up and solve these triangles:

- In triangle ABC, A is opposite $a$, B is opposite $b$, and C is opposite $c$. Always draw and label triangles.
- The three angles of a triangle add up to 180°.
- The longest side of a triangle is always opposite the largest angle.
- Always use the given information as much as possible.
- Never round until the final answer for an angle or a side. Include units in answers if possible.
- In general round to the nearest degree for angles and the nearest tenth for sides.
- **Show the equations you use.**

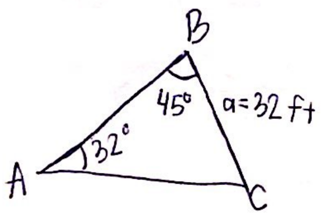
**Law of Sines:** If ABC is a triangle with sides $a$, $b$, and $c$, then:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

***Use to find sides*** ***Use to find angles***

*Examples.*

1. (AAS) For the triangle ABC, $A = 30°$, $B = 45°$ and $a = 32$ feet. Find the remaining angle and sides.




Angle C

$32° + 45° + C = 180°$

$77° + C = 180°$

$-77° \quad -77°$

$C = 103°$

Side c

$\frac{c}{\sin 103°} = \frac{32}{\sin 32}$

$\frac{c \sin 32}{\sin 32} = \frac{32 \cdot \sin 103}{\sin 32}$

$c = 58.8$ ft

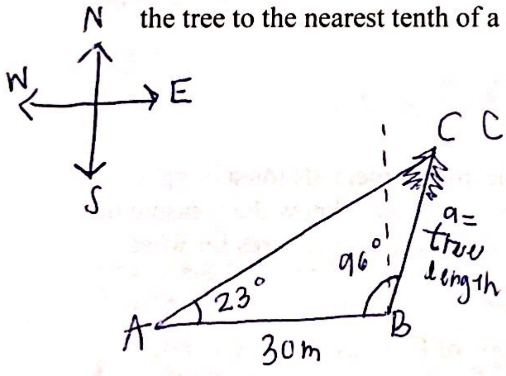
Side b

$\frac{b}{\sin 45°} = \frac{32}{\sin 32}$

$\frac{b \sin \cancel{32}}{\cancel{\sin} 32} = \frac{32 \sin 45}{\sin 32}$

$b = 42.7$ ft

2. (ASA) Because of prevailing winds, a tree grew so that it was leaning 6° east of vertical. At a point 30 meters west of the base of the tree, the angle of elevation to the top of the tree is 23°. Find the length of the tree to the nearest tenth of a meter.




Angle C

$C + 23° + 96° = 180°$

$C + 119° = 180°$

$-119 \quad -119$

$C = 61°$

Side a (tree length)

$\frac{a}{\sin 23} = \frac{30\text{m}}{\sin 61°}$

$\frac{a \sin 61°}{\sin 61} = \frac{30 \sin 23}{\sin 61}$

$a = 13.4$

tree length $= 13.4$ m

In Examples 1 and 2 (Case 1), two angles and one side determine exactly one triangle. However, if two sides and one opposite angle are given (Case 2), three possible situations can occur:

1. No triangle exists. 2. Exactly one triangle exists. 3. Two different triangles exist.

You must decide which situation applies to be able to successfully solve the triangle(s).

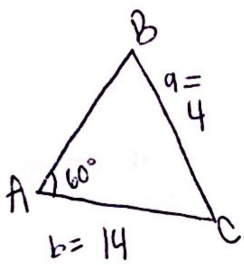
Here is a suggested method (recipe) for solving triangles given **SSA.**

Solve for the second angle using the Law of Sines, and see which of the following **3 cases** applies.

- If there is **no triangle**, your calculator will give you an error message, because you will be trying to do an inverse sine for a ratio bigger than 1.
- If there is **one triangle**, the angle that you find using the Law of Sines will be smaller than the given angle, or the angle that you find will be exactly 90 degrees.
- If there are **two triangles**, the angle that you find will be larger than the given angle (but not 90°). To find the second angle for the second triangle, subtract the angle value found on the calculator from 180°. Draw both triangles.

*Examples.*

3. $A = 60^\circ$, $a = 4$ ft, and $b = 14$ ft. Find the remaining sides and angles for any possible triangles.




Angle B:

$\frac{\sin B}{14} = \frac{\sin 60^\circ}{4}$

$\frac{4 \cdot \sin B}{4} = \frac{14 \cdot \sin 60^\circ}{4}$

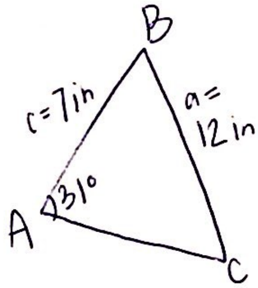
$\sin B = 3.031\ldots$ (keep in calc)

$\sin^{-1}(3.031\ldots) = B$

$\Rightarrow$ Error b/c 3 is larger than 1.

$\boxed{\text{No triangle Exists}}$

4. $A = 31^\circ$ $a = 12$ in, and $c = 7$ in. Find the remaining sides and angles for any possible triangles.




Angle C:

$\frac{\sin C}{7} = \frac{\sin 31}{12}$

$\frac{\cancel{12}\sin C}{\cancel{12}} = \frac{7 \sin 31}{12}$

$\sin C = .3004\ldots$

$C = \sin^{-1}(.3004\ldots)$

$\boxed{C = 17^\circ}$

angle C < angle A $\Rightarrow$ one triangle

angle found    angle given

Angle B: $31 + 17 + B = 180^\circ$

$\boxed{B = 132^\circ}$

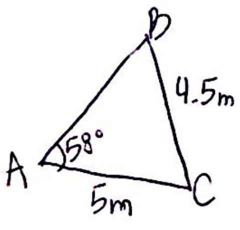
Side b:

$\frac{b}{\sin 132} = \frac{12}{\sin 31}$

$\frac{b \cdot \sin 31}{\sin 31} = \frac{12 \sin 132}{\sin 31}$

$\boxed{b = 17.3 \text{ in}}$

5. $A = 58^\circ$ $a = 4.5$ m, and $b = 5$ m. Find the remaining sides and angles for any possible triangles.




Angle B:

$\frac{\sin B}{5} = \frac{\sin 58}{4.5}$

$\frac{4.\cancel{5} \cdot \sin B}{\cancel{4.5}} = \frac{5 \sin 58}{4.5}$

$\sin B = .9422\ldots$

$B = \sin^{-1}(.9422\ldots)$ $\boxed{B = 70^\circ}$ angle found

Angle C:

$58^\circ + 70^\circ + C = 180$

$\boxed{C = 52^\circ}$

Side c:

$\frac{c}{\sin 52} = \frac{4.5}{\sin 58^\circ}$

$\frac{c \cancel{\sin 58}}{\cancel{\sin 58}} = \frac{4.5 \sin 52}{\sin 58}$

$\boxed{c = 4.2 \text{m}}$

Ang B > Ang A -> 2 triangles exits

angle given

2nd triangle:

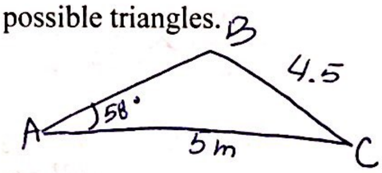
Angle B:

$180^\circ - 70^\circ = B$

$\boxed{B = 110^\circ}$

Angle C: $C = 180 - 58 - 110$

$\boxed{C = 12^\circ}$




Side c

$\frac{c}{\sin 12^\circ} = \frac{4.5}{\sin 58^\circ}$

$\frac{c \cdot \cancel{\sin 58^\circ}}{\cancel{\sin 58^\circ}} = \frac{4.5 \sin 12^\circ}{\sin 58^\circ}$

$\boxed{c = 1.1 \text{m}}$

Assignment 7.3

*Day 4*

## Law of Cosines

Last time you learned how to use the Law of Sines to solve triangles of the forms AAS, ASA, or SSA. This section discusses the use of the **Law of Cosines** to solve triangles of the forms SAS or SSS. Although the Law of Cosines is harder to use than the Law of Sines, there are no ambiguous cases.

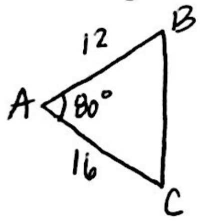
**Law of Cosines:**

| SAS (use to find a side) | | SSS (use to find the largest angle) |
|---|---|---|
| $a^2 = b^2 + c^2 - 2bc \cos A$ | or | $\cos A = \frac{b^2+c^2-a^2}{2bc}$ |
| $b^2 = a^2 + c^2 - 2ac \cos B$ | or | $\cos B = \frac{a^2+c^2-b^2}{2ac}$ |
| $c^2 = a^2 + b^2 - 2ab \cos C$ | or | $\cos C = \frac{a^2+b^2-c^2}{2ab}$ |

Use the Law of Cosines once. Then use the Law of Sines to finish solving the triangle. But realize, the Law of Sines can never be used to find an obtuse angle, it will only give acute angles. So make sure the angle you are solving for is less than 90°. i.e. solve for the smaller angle first!

*Examples.*

1. (SAS) Given a triangle with $A = 80^\circ, c = 12$ inches, and $b = 16$ inches, find the remaining sides and angles.




Side A:

$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 16^2 + 12^2 - 2(16)(12)\cos 80^\circ$

$a^2 = 333.3\ldots$

$\boxed{a = 18.3 \text{ in}}$

Angle B:

$180 - 80^\circ - 40^\circ$

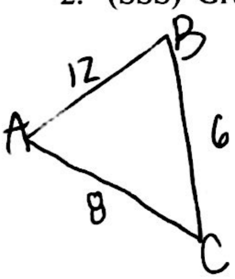
$\boxed{B = 60^\circ}$

Angle C:

$\frac{\sin C}{12} \times \frac{\sin 80}{18.3}$ $\quad \frac{\cancel{18.3} * \sin C}{\cancel{18.3}} = \frac{12 \cdot \sin 80}{18.3}$ $\quad \boxed{C = 40^\circ}$

$\sin C = .6457\ldots \quad C = \sin^{-1}(.645\ldots)$

2. (SSS) Given a triangle with $a = 6$ feet, $b = 8$ feet, and $c = 12$ feet, find $A$, $B$, and $C$.




Angle C

$\cos C = \frac{6^2 + 8^2 - 12^2}{2(6)(8)}$

$\cos C = -.45833\ldots$

$\boxed{C = 117^\circ}$

Angle A:

$\frac{\sin A}{6} \times \frac{\sin 117}{12}$

$\frac{\cancel{12} \sin A}{\cancel{12}} = \frac{6 \sin 117}{12}$

$\sin A = .4455\ldots$

$A = \sin^{-1}(.4455)$

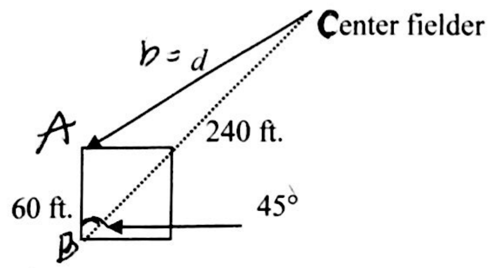
$\boxed{A = 26^\circ}$

Angle B:

$180 - 117 - 26 = 37$

$\boxed{B = 37^\circ}$

3. In a softball game, a batter hits the ball to center field. The center fielder then throws the ball to third base, as shown. The distance from the center fielder to home plate is 240 feet, and the distance between the bases is 60 feet. How far did the center fielder throw the ball?




$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 240^2 + 60^2 - 2(240)(60)\cos 45°$

$b^2 = 40835.3$

$b = 202.1$ $\boxed{202.1 \text{ ft}}$

SAS

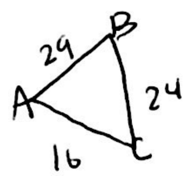
Deciding whether to start to solve a triangle by use of the Law of Sines or Law of Cosines will be critical to your success. You should always draw an accurate sketch before doing your work.

*Examples.* Determine whether the Law of Sines or Law of Cosines should be used to start to solve each of the following triangles. You do not need to actually solve them.

4. $a = 24$ feet, $b = 16$ feet, $c = 29$ feet

Given : SSS

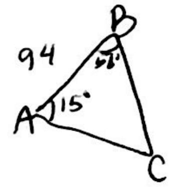
Use: Cos




5. $A = 15^\circ, B = 58^\circ$, $c = 94$ inches

Given: ASA

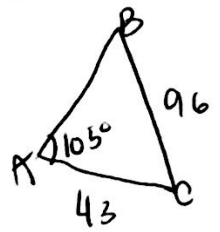
Use: Sin




6. $a = 96$ meters, $b = 43$ meters, $A = 105^\circ$

Given: SSA

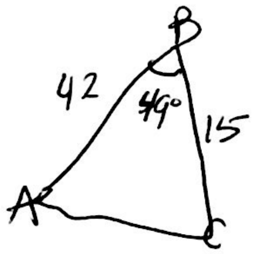
Use: Sin




7. $a = 15$ yards, $c = 42$ yards, $B = 49^\circ$

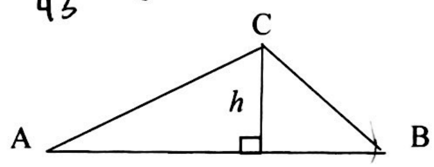
Given: SAS

Use: Cos




## Area of a Triangle

Given triangle ABC as shown:



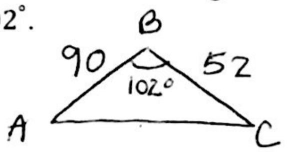

The area of triangle ABC is $area = \frac{1}{2}ch$, where $h$ is the height of the triangle. But what if you don't know the value of $h$?

There are two formulas for finding areas of oblique triangles:

| SAS | SSS |
|---|---|
| $Area = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$ | $Area = \sqrt{s(s-a)(s-b)(s-c)}$ Where $s = \frac{a+b+c}{2}$ |

*Examples.*

8. Find the area of a triangular lot if two sides have lengths 90 meters and 52 meters, and the included angle (inbetween) is 102°.



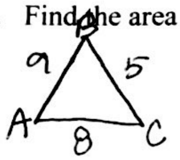

$a = 52$
$c = 90$
$B = 102$

$\text{area} = \frac{1}{2} ac \sin B$

$\text{area} = \left(\frac{1}{2} * 52 * 90 * \sin 102\right)$

$\boxed{\text{area} = 2288.9 m^2}$

9. Find the area of the triangle having sides which measure 5 feet, 8 feet, and 9 feet.



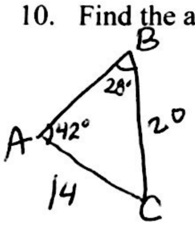

$a = 5 \quad b = 8 \quad c = 9$

$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \frac{a+b+c}{2} = \frac{5+8+9}{2} = 11$

$\text{area} = \sqrt{11(11-5)(11-8)(11-9)} = \boxed{19.9 ft^2}$

10. Find the area of triangle ABC given $A = 42°$, $B = 28°$ and $b = 14$ inches.




$\frac{1}{2} ab \sin C$ or $\frac{1}{2} bc \sin A$

$\text{area} = \frac{1}{2} * 20 * 14 * \sin 110$

$= \boxed{131.6 in^2}$

side a:

$\frac{a}{\sin 42} \times \frac{14}{\sin 28}$ $\quad a = 20$

Angle $C = 180 - 28 - 42$
$= 110°$

Assignment 7.4

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*Day 5*

**Unit 7 Review**

Assignment 7.5

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*Day 6*

**Unit 7 Test** ***All late/absent assignments due for Unit 7***

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